If $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ verify cayley's hamilton theorem hence find A^{-1}

Sol: Given matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

We know that the characteristic equation of A is $|A-\lambda I|$

$$\Rightarrow \left| \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 1-\lambda\left[(1-\lambda)(2-\lambda)-1\right] + 1\left[0-2\right] + 0 = 0$$

$$\Rightarrow 1-\lambda\left[2-\lambda-2\lambda+\lambda^2-1\right] - 2 = 0$$

$$\Rightarrow 1-\lambda\left[\lambda^2-3\lambda+1\right] - 2 = 0$$

$$\Rightarrow \lambda^2-3\lambda+1-\lambda^3+3\lambda^2-\lambda-2 = 0$$

$$\Rightarrow \lambda^3+4\lambda^2-4\lambda-1 = 0$$

$$\Rightarrow \lambda^3-4\lambda^2+4\lambda+1 = 0 \rightarrow (1)$$

Equ (1) is the characteristic equation of matrix A.

To verify cayley hamilton theorem put $\lambda = A$ in equ (1)

$$\Rightarrow A^3-4A^2+4A+1 = 0 \rightarrow (2)$$

To find A^2:

$\therefore A^2 = A \cdot A$

$$A^2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1-0+0 & -1-1+0 & 0-1+0 \\ 0+0+2 & 0+1+1 & 0+1+2 \\ 2+0+4 & -2+1+2 & 0+1+4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix}$$

To find A^3:

$\therefore A^3 = A^2 \cdot A$

$$A^3 = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1\times1-2\times0-1\times2 & 1\times-1-2\times1-1\times1 & 1\times0-2\times1-1\times2 \\ 2\times1+2\times0+3\times2 & 2\times-1+2\times1+3\times1 & 2\times0+2\times1+3\times2 \\ 6\times1+1\times0+5\times2 & 6\times-1+1\times1+5\times1 & 6\times0+1\times1+5\times2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1-2 & -1-2-1 & -2-2 \\ 2+6 & -2+2+3 & 2+6 \\ 6+10 & -6+1+5 & 1+10 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & -4 & -4 \\ 8 & 3 & 8 \\ 16 & 0 & 11 \end{bmatrix}$$

equ ② $\rightarrow$ $\left\{ \begin{bmatrix} 1 & 4 & -4 \\ 8 & 3 & 8 \\ 16 & 0 & 11 \end{bmatrix} - 4 \begin{bmatrix} 1 & -2 & 1 \\ 2 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix} + 4 \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right.$

$\Rightarrow \left\{ \begin{bmatrix} -1 & 4 & -4 \\ 8 & 3 & 8 \\ 16 & 0 & 11 \end{bmatrix} - \begin{bmatrix} 4 & -8 & -4 \\ 8 & 8 & 12 \\ 24 & 4 & 20 \end{bmatrix} + \begin{bmatrix} 4 & -4 & 0 \\ 0 & 4 & 4 \\ 8 & 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} = 0$

$\Rightarrow \begin{bmatrix} -1-4+4+1 & -4+8-4+0 & -4+4+0+0 \\ 8-8+0+0 & 3-8+4+1 & 8-12+4+0 \\ 16-24+8+0 & 0-4+4+0 & 11-20+8+1 \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$

∴ cayley hamilton theorem is verified.

The given matrix A satisfies its characteristic equation.

The given matrix satisfies cayley hamilton theorem

To find A Inverse :

operate A^{-1} on equ ②

$\Rightarrow A^{-1} \cdot A^3 - 4A^2 A^{-1} + 4AA^{-1} + 1 = 0$

$\Rightarrow A^2 - 4A + 4I + A^{-1} = 0$

$\Rightarrow$ The given set is linearly independent. which is contradiction to the given condition 's' is linearly dependent.

If $k \neq 1$

$\Rightarrow 2 \leq k \leq n$

From (1) $\Rightarrow a_k \alpha_k = (-a_1)\alpha_1 + (-a_2)\alpha_2 + \dots + (-a_{k+1})\alpha_{k+1}$

$$\Rightarrow \alpha_k = \left(\frac{-a_1}{a_k}\right)\alpha_1 + \left(\frac{-a_2}{a_k}\right)\alpha_2 + \dots + \left(\frac{-a_{k+1}}{a_k}\right)\alpha_{k+1}$$

$\therefore$ α_k can be expressed as a linear combination of its preciding vectors

Hence the Neccessary Condition.

Part II : Sufficient condition

If sum vector $\alpha_k \in S$, $2 \leq k \leq n$ can be expressed as linear combination of its preciding vectors then 's' is L.D.

Given, $\alpha_k \in S$, $2 \leq k \leq n$ can be expressed as a linear combination of its preciding vector.

$\Rightarrow \exists$ a $b_1, b_2 \dots b_k \in F$

$$\alpha_k = b_1\alpha_1 + b_2\alpha_2 + \dots + b_{k-1}\alpha_{k-1}$$

$$\Rightarrow b_1\alpha_1 + b_2\alpha_2 + \dots + b_{k-1}\alpha_{k-1} + (-1)\alpha_k = \bar{0}$$

$\Rightarrow \{\alpha_1, \alpha_2 \dots \alpha_{k-1}, \alpha_k\}$ is linearly dependent.

$\Rightarrow \{\alpha_1, \alpha_2 \dots \alpha_{k-1} \dots \alpha_n\}$ is linearly dependent

Hence the sufficient condition.

Hence the proof

Theorem 16 :

Let $V(F)$ be a vector space, if $\alpha_1, \alpha_2 \dots \alpha_n$ are Non-zero vectors of V then, either they are linearly independent or some α_k, $2 \leq k \leq n$ is a linear combination of the preciding onces $\alpha_1, \alpha_2 \dots \alpha_{k-1}$.

Proof : Write the Neccessary part of the above theorem.

Imp

Theorem 17 : Let $V(F)$ be a vector space and $\alpha_1, \alpha_2 \dots \alpha_n$ if some α_k, $2 \leq k \leq n$ is a linear combination of the preciding onces then $\alpha_1, \alpha_2 \dots \alpha_n$ are L.D.

Proof : Write the sufficient part of theorem 15.

To Find (i):

If $(a,b,c,d) \in W_1 \Rightarrow b-2c+d=0$
$\Rightarrow b=2c-d$

$\therefore (a,b,c,d) = (a, 2c-d, c, d)$
$= a(1,0,0,0) + c(0,2,1,0) + d(0,-1,0,1)$

$\therefore$ The set $\{(1,0,0,0)\ (0,2,1,0)\ (0,-1,0,1)\}$ is linearly indep and form the basis of W_1.

$\therefore \boxed{\dim(W_1) = 3}$

To Find (ii):

If $(a,b,c,d) \in W_2 \Rightarrow a=d,\ b=2c$

$(a,b,c,d) = (d, 2c, c, d)$
$= c(0,2,1,0) + d(1,0,0,1)$

$\therefore$ The set $\{(0,2,1,0)\ (1,0,0,1)\}$ is linearly independent form the basis of W_2.

$\therefore \boxed{\dim(W_2) = 2}$

To Find (iii):

If $(a,b,c,d) \in W_1 \cap W_2$
$\Rightarrow b-2c+d=0 \longrightarrow (1)$
and $a=d,\ b=2c$

put $b=2c$ in equ (1) $\Rightarrow \cancel{2c}-\cancel{2c}+d=0$
$\Rightarrow d=0$
$\Rightarrow a=0$
$\therefore (a,b,c,d) = (0,2c,c,0)$
$= c(0,2,1,0)$

$\therefore$ The set $\{(0,2,1,0)\}$ is linearly indepent and forms the ba of $W_1 \cap W_2$.

$\therefore \boxed{\dim(W_1 \cap W_2) = 1}$

We know that $\dim(W_1+W_2) = \dim(W_1)+\dim(W_2)-\dim($
$= 3+2-1$
$= 4$

Q) To prove $\left(\frac{V}{W}, +\right)$ is a commutative group :

i) Commutative axiom :

$\forall\ \alpha, \beta \in V \Rightarrow \alpha+\beta \in V$

If $\alpha, \beta \in V \Rightarrow W+\alpha,\ W+\beta \in \frac{V}{W}$

Then $\alpha+\beta$ $(W+\alpha)+(W+\beta) = W+(\alpha+\beta) \in \frac{V}{W}$ w.r.t.

$\therefore \frac{V}{W}$ is closed under the given composition addition

ii) Associative axiom :

$\forall\ \alpha, \beta, \gamma \in V \Rightarrow \alpha+(\beta+\gamma) = (\alpha+\beta)+\gamma$

If $\alpha, \beta, \gamma \in V \Rightarrow (W+\alpha), (W+\beta), (W+\gamma) \in \frac{V}{W}$

Consider,

$$(W+\alpha)+[(W+\beta)+(W+\gamma)] = (W+\alpha)+[W+(\beta+\gamma)]$$
$$= W+[\alpha+(\beta+\gamma)]$$
$$= W+[(\alpha+\beta)+\gamma]$$
$$= W+[(\alpha+\beta)+(W+\gamma)]$$
$$= [W+(\alpha+\beta)]+(W+\gamma)$$

i.e., $(W+\alpha)+[(W+\beta)+(W+\gamma)] = [(W+\alpha)+(W+\beta)]+(W+\gamma)$

$\therefore$ Composition '+' is associative in $\frac{V}{W}$

iii) Identity axiom :

$\forall\ \alpha \in V \Rightarrow \exists$ a vector $\bar{0} \in V$ such that

$a+0 = a = 0+a$.

$\alpha+\bar{0} = \alpha = \bar{0}+\alpha$

If $\alpha, \bar{0} \in V \Rightarrow W+\alpha,\ W+\bar{0} \in \frac{V}{W}$

Consider, $(W+\alpha)+(W+\bar{0}) = W+(\alpha+\bar{0})$

$= W+\alpha$

Similarly, $(W+\bar{0})+(W+\alpha) = W+\alpha$.

$\therefore (W+\bar{0}) = W$ is the identity element in $\frac{V}{W}$

iv) Inverse axiom :

$\forall\ \alpha \in V \Rightarrow \exists$ a vector $-\alpha \in V$ such that zero vector

Since, $1\cdot 0 + 0\alpha_2 + 0\alpha_3 + \dots + 0\alpha_n = 0$

Here $1, 0, 0 \dots 0 \in F$ not all zeros

$\Rightarrow$ 'S' is linearly dependent.

Hence the proof

Theorem 12 :

Let $V(F)$ be a vector space. If two vector in V are linearly dependent then one of them is a scalar multiple of the other.

proof : Given $V(F)$ be a vector space.

Let $S = \{\alpha, \beta\}$ are linearly dependent.

Let there exist scalars $a_1, a_2 \in F$ not all zero

such that $a_1\alpha + a_2\beta = 0$

let $a_1 \neq 0$ then $a_1\alpha + a_2\beta = 0$

$\Rightarrow a_1\alpha = -a_2\beta$

$\Rightarrow \alpha = \left(\frac{a_2}{a_1}\right)\beta$

$\Rightarrow$ 'α' is the scalar multiple of β

Similarly, if $a_2 \neq 0 \Rightarrow \beta$ is the scalar multiple of α

Hence the proof

Theorem 13 :

Let $V(F)$ be a vector space then every super set of a LD set of V is linearly dependent

Proof : Given $V(F)$ be a vector space

To prove that every super set of a Linearly dependent set of V is linearly dependent

Let $S = \{\alpha_1, \alpha_2 \dots \alpha_n\}$ is linearly dependent

let there exist $a_1, a_2 \dots a_n \in F$ not all zero

such that $a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = \bar{0} \rightarrow ①$

Let $S^1 = \{\alpha_1, \alpha_2 \dots \alpha_n, \beta_1, \beta_2 \dots \beta_m\}$ be the super set of S

$\therefore$ equ ① $\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = \bar{0}$

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n + 0\beta_1 + 0\beta_2 + \dots + 0\beta_m = 0$

Where scalars are $a_1, a_2 \dots a_n, 0, 0 \dots 0 \in F$ Not all zero

$\Rightarrow S^1$ is linearly dependent.

Hence the proof.

Theorem 14: Let $V(F)$ is a vector space, a subset of a linearly independent set of 'V' is linearly independent.

proof: Given $V(F)$ is a vector space

To prove that a sub set of a linearly independent set, of 'V' is L.I

let $S = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ is L.I

Let there exist scalar $a_1, a_2 \ldots a_n \in F$ such that

$a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = 0$

$\Rightarrow a_1 = 0, a_2 = 0 \ldots a_n = 0$

let $S' = \{\alpha_1, \alpha_2 \ldots \alpha_m\}$ $(m<n)$

To S.T S' is L.I :

Let there exist scalars $a_1, a_2 \ldots a_m \in F$

such that $a_1\alpha_1 + a_2\alpha_2 + \ldots + a_m\alpha_m = \bar{0}$

$\rightarrow a_1 = 0, a_2 = 0, \ldots a_m = 0$

$\Rightarrow S'$ is linearly independent.

Hence the proof.

Imp

Theorem 15:

$V(F)$ be a vector space and $S = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ is a finite sub set of Non-zero vectors of $V(F)$. Then 'S' is linearly dependent if and only iff sum vector $\alpha_k \in S$, $2 \leq k \leq n$ can be expressed as a linear combination of its preciding vectors.

proof: Given $S = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ is a Non-zero sub set of vectors in $V(F)$.

part I :

If 'S' is linearly dependent, then S.T there exist some vectors $\alpha_k \in S$, $2 \leq k \leq n$ can be expressed as linear combination of its preciding vectors.

Given, 'S' is linearly dependent

let 'k' be the largest suffix such that $a_k \neq 0$

$\therefore a_1\alpha_1 + a_2\alpha_2 + \ldots + a_k\alpha_k + 0\alpha_{k+1} + \ldots + 0\alpha_n = \bar{0}$

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \ldots + a_{k-1}\alpha_{k-1} + a_k\alpha_k = \bar{0} \rightarrow (1)$

If $k=1$

Then $a_1\alpha_1 = \bar{0} \Rightarrow a_1 = 0$ $[\because \alpha_1 \neq \bar{0}]$

$\Rightarrow$ The given set is linearly independent. which is contradiction to the given condition 's' is linearly dependent.

<u>If $k \neq 1$</u>

$\Rightarrow 2 \le k \le n$

From (1) $\Rightarrow a_k \alpha_k = (-a_1)\alpha_1 + (-a_2)\alpha_2 + \dots + (-a_{k+1})\alpha_{k+1}$

$\Rightarrow \alpha_k = \left(\frac{-a_1}{a_k}\right)\alpha_1 + \left(\frac{-a_2}{a_k}\right)\alpha_2 + \dots + \left(\frac{-a_{k+1}}{a_k}\right)\alpha_{k+1}$

$\therefore \alpha_k$ can be expressed as a linear combination of its preciding vectors

Hence the Neccessary Condition.

<u>Part II</u> : <u>Sufficient condition</u>

If sum vector $\alpha_k \in S$, $2 \le k \le n$ can be expressed as a linear combination of its preciding vectors then 's' is L.D.

Given, $\alpha_k \in S$, $2 \le k \le n$ can be expressed as a linear combination of its preciding vector.

$\Rightarrow \exists$ a $b_1, b_2 \dots b_k \in F$

$\alpha_k = b_1\alpha_1 + b_2\alpha_2 + \dots + b_{k-1}\alpha_{k-1}$

$\Rightarrow b_1\alpha_1 + b_2\alpha_2 + \dots + b_{k-1}\alpha_{k-1} + (-1)\alpha_k = \bar{0}$

$\Rightarrow \{\alpha_1, \alpha_2 \dots \alpha_{k-1}, \alpha_k\}$ is linearly dependent.

$\Rightarrow \{\alpha_1, \alpha_2 \dots \alpha_{k-1} \dots \alpha_n\}$ is linearly dependent

Hence the sufficient condition.

Hence the proof

<u>Theorem 16</u> :

Let $V(F)$ be a vector space, if $\alpha_1, \alpha_2 \dots \alpha_n$ are Non-zero vectors of V then, either they are linearly independent or some α_k, $2 \le k \le n$ is a linear combination of the preciding onces $\alpha_1, \alpha_2 \dots \alpha_{k-1}$.

Proof : Write the Neccessary part of the above theorem.

Imp

<u>Theorem 17</u> : Let $V(F)$ be a vector space and $\alpha_1, \alpha_2 \dots \alpha_n$ if some α_k, $2 \le k \le n$ is a linear combination of the preciding onces then $\alpha_1, \alpha_2 \dots \alpha_n$ are L.D.

<u>Proof</u> : Write the sufficient part of theorem (15).

Theor

Let $V(F)$ be a vetor space and $S = \{\alpha_1, \alpha_2 \ldots \alpha_n\} \subseteq V$ If some α_i, $2 \le i \le n$ is a linear combination of the preciding once $\alpha_1, \alpha_2 \ldots \alpha_{i-1}$ then $L(S) = L(S')$, where $S' = \{\alpha_1, \alpha_2 \ldots \alpha_{i-1}, \alpha_{i+1} \ldots \alpha_n\}$.

Proof: Given that $V(F)$ be a vetor space.

And $S = \{\alpha_1, \alpha_2 \ldots \alpha_{i-1}, \alpha_i, \alpha_{i+1} \ldots \alpha_n\}$

also given $S' = \{\alpha_1, \alpha_2 \ldots \alpha_{i-1}, \alpha_{i+1} \ldots \alpha_n\}$

clearly $S' \subseteq S$

$\Rightarrow L(S') \subset L(S) \longrightarrow (1)$

Now, To prove that $L(S) \subseteq L(S')$

let $\alpha \in L(S)$

$\Rightarrow$ 'α' can be expressed as linear combination of vectors of 'S'

$\Rightarrow \alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_{i-1}\alpha_{i-1} + a_i\alpha_i + a_i\alpha_i + \ldots + a_{i+1}\alpha_{i+1} + \ldots + a_n\alpha_n$ (2)

given that α_i is the linear combination of $\alpha_1, \alpha_2 \ldots \alpha_{i-1}$

$\Rightarrow \alpha_i = b_1\alpha_1 + b_2\alpha_2 + \ldots + b_{i-1}\alpha_{i-1}$

From (2) $\Rightarrow \alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_{i-1}\alpha_{i-1} + a_i(b_1\alpha_1 + b_2\alpha_2 + \ldots + b_{i-1}\alpha_{i-1}) + a_{i+1}\alpha_{i+1} + \ldots + a_n\alpha_n$

$\Rightarrow \alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_{i-1}\alpha_{i-1} + a_ib_1\alpha_1 + a_ib_2\alpha_2 + \ldots + a_ib_{i-1}\alpha_{i-1} + a_{i+1}\alpha_{i+1} + \ldots + a_n\alpha_n$

$\Rightarrow \alpha = \alpha_1(a_1 + a_ib_1) + \alpha_2(a_2 + a_ib_2) + \ldots + \alpha_{i-1}(a_{i-1} + a_ib_{i-1}) + a_{i+1}\alpha_{i+1} + \ldots + a_n\alpha_n$

i.e., 'α' can be expressed as a linear combination of vectors of $\alpha_1, \alpha_2 \ldots \alpha_{i-1}, \alpha_{i+1} \ldots \alpha_n$

$\Rightarrow \alpha$ = linear combination of vectors of S'.

$\Rightarrow \alpha \in L(S')$.

$\therefore$ If $\alpha \in L(S) \Rightarrow \alpha \in L(S')$.

$\Rightarrow L(S) \subseteq L(S') \longrightarrow (3)$

$\therefore$ From (1) & (3) $\Rightarrow L(S') \subseteq L(S)$, $L(S) \subseteq L(S')$

$\Rightarrow L(S) = L(S')$.

Hence the proof.

Vector space - II

Define basis of a vector space:

A Non-empty sub set 's' of a vector space $V(F)$ is set to be "a basis", if i) 's' is linearly independent

ii) $L(S) = V$, ie; every vector in 'v' can be expressed as a linear combination of vectors in 's'.

Note:

1) The set $S = \{e_1, e_2 \ldots e_n\}$ be the standard basis for the vector space $V_n(F)$

2) The set $S = \{(1,0,0)\ (0,1,0)\ (0,0,1)\}$ be the standard basis for the vector spac $V_3(F)$

Finite dimensional vector space:

A vector space $V(F)$ is said to "Finite dimensional" (or) "finitely generated". If there exist a finite sub set of 'v' such that $L(S) = V$.

Imp

Theroem 19:

If $V(F)$ is a finite dimensional vector space, then there exist a basis set of V.

(or)

Proof:

Every finite dimensional vector space has a basis.

Proof: Given $V(F)$ is a finite dimensional vector space.

let $S = \{\alpha_1, \alpha_2 \ldots \alpha_{k-1}, \alpha_k, \alpha_{k+1}, \ldots \alpha_n\}$ be any Non-Empty sub set of $V(F)$.

ie, $L(S) = V$, since 'v' is a vector space.

Now, We S.T 's' is linearly independent:

If 's' is linearly independent then 's' it self is a basis of $V(F)$.

If 's' is linearlyindependent then linearly dependent We know that by one of the theorem there exist a vector α_k in 's' can be expressed as a linear combination of it's preceding vectors.

$\therefore$ There exist scalars $a_1, a_2 \ldots a_{k-1} \in F$ such that

$$\alpha_k = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_{k-1}\alpha_{k-1}$$

Remove the vector α_K from the set 's', we get a set s_1

$s_1 = \{\alpha_1, \alpha_2 \ldots \alpha_{K-1}, \alpha_{K+1} \ldots \alpha_n\}$

If 's_1' is linearly independent then 's_1' it self is a basis of $V(F)$

If s_1 is linearly dependent then proceed of above, after sum finite number of steps we get a set containg only one vector say $S_K = \{\alpha_1\}$

WKT, S_K is linearly independent [$\because a_1\alpha_1 = 0 \Rightarrow a_1 = 0$]

$\therefore$ 'S_K' is the basis of $V(F)$.

$\therefore$ "Every finite dimensional vector space has a basis"

Hence the proof

Problems :

S.T, the set $S = \{(1,0,-2)\ (1,2,1)\ (0,-3,2)\}$ forms a basis of a vector space $V_3(R)$.

Sol: Given, $S = \{\alpha_1, \alpha_2, \alpha_3\}$

Where $\alpha_1 = (1,0,-2)$

$\alpha_2 = (1,2,1)$

$\alpha_3 = (0,-3,2)$

To prove 's' is a basis, we S.T

i) 's' is linearly independent :

Let there exist scalars $a_1, a_2, a_3 \in R$ such that

$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \bar{0}$

$\Rightarrow a_1(1,0,-2), + a_2(1,2,1) + a_3(0,-3,2) = (0,0,0)$

$\Rightarrow a_1 + a_2 + 0$. $\quad 2a_2 - 3a_3 = 0 \rightarrow (2)$ $-2a_1 + a_2 + 2a_3 = 0 \rightarrow (3)$

$\Rightarrow a_1 + a_2 = 0 \rightarrow (1)$

The above equ's can be written in matrix form as follows

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ co-efficient matrix $= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -2 & 1 & 2 \end{bmatrix}$ $R_3 \Rightarrow 2R_1 + R_3$

$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ 0 & 3 & 2 \end{bmatrix}$ $R_3 \Rightarrow 3R_2 - 2R_3$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ 0 & 0 & -13 \end{bmatrix}$$

The above matrix is an echlon form

$\therefore$ Rank of A - $\rho(A)$ = No of Non-zero rows in the echlon form = 3

and Ran No. of variable = 3

No of variable = No of Non-zero rows

$\therefore$ The given set 's' is linearly independent

ii) $\underline{L(S)} = \underline{V_3(R)}$:

Let $(a,b,c) \in V_3(R)$ be any vector in $V_3(R)$

Let there exist scalars $a_1, a_2, a_3 \in R$ such that

$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = (a,b,c) \longrightarrow$ (a)

$\Rightarrow a_1(1,0,-2) + a_2(1,2,1) + a_3(0,-3,2) = (a,b,c)$

$\Rightarrow \therefore a_1 + a_2 = a$

$2a_2 - 3a_3 = b$

$-2a_1 + a_2 + 2a_3 = c$

The above equation can be written in matrix formas follows.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\therefore$ Argumented Matrix, $\begin{bmatrix} 1 & 1 & 0 & : a \\ 0 & 2 & -3 & : b \\ -2 & 1 & 2 & : c \end{bmatrix}$ $R_3 \to 2R_1 + R_3$

2a+c

$$\sim \begin{bmatrix} 1 & 1 & 0 & : a \\ 0 & 2 & -3 & : b \\ 0 & 3 & 2 & : 2a+c \end{bmatrix} R_3 \Rightarrow 3R_2 - 2R_3$$

3b - 4a+2c

$$\sim \begin{bmatrix} 1 & 1 & 0 & : a \\ 0 & 2 & -3 & : b \\ 0 & 0 & -13 & : 3b-4a-2c \end{bmatrix}$$

The above matrix can be written in Equation form as follow

$a_1 + a_2 + 0 \cdot a_3 = a \longrightarrow$ (4)

$0 \cdot a_1 + 2a_2 - 3a_3 = b \longrightarrow$ (5)

$$0a_1 + 0a_2 - 13a_3 = 3b - 4a - 2c$$ ⑤ $a_1 = \frac{a-b+2c}{2}$, $a_2 = \frac{a-b+2c}{2}$, $a_3 = \frac{2b-a}{2}$.

$$\rightarrow a_3 = \frac{-1}{13}(3b-4a-2c)$$

$$\Rightarrow a_3 = \frac{4a-3b+2c}{13}$$

put a_3 values in equ ⑤ $\Rightarrow 2a_2 - 3\left(\frac{4a-3b+2c}{13}\right) = b$

$$\Rightarrow 2a_2 = \frac{3}{13}(4a-3b+2c)+b$$

$$\Rightarrow 2a_2 = \frac{12a-9b+6c+13b}{13}$$

$$\Rightarrow a_2 = \frac{12a+4b+6c}{26}$$

$$\Rightarrow a_2 = \frac{2(6a+2b+3c)}{26}$$

$$\Rightarrow a_2 = \frac{6a+2b+3c}{13}.$$

put a_2, a_3 values in equ ④ ④ $a_1 = \frac{a+c}{2}$, $a_2 = \frac{2b-a-c}{2}$, $a_3 = b-c$

$$a_1 + \frac{6a+2b+3c}{13} = a$$

$$a_1 = \frac{-6a-2b-3c}{13} + a$$

$$a_1 = \frac{-6a-2b-3c+13a}{13}$$

$$a_1 = \frac{7a-2b-3c}{13}$$

put a_1, a_2, a_3 values in equ ⓐ

$$(a,b,c) = \frac{1}{13}(7a-2b-3c)(1,0,2) + \frac{1}{13}(6a+2b+3c)(1,2,1) + \frac{1}{13}(4a-3b+2c)(0,-3,2)$$

$\therefore$'s' is linearly independent and $L(S) = V$

$\therefore$ 'S' is a basis of $V_3(R)$. $a_1 = \frac{1}{9}(-2a+4b+3c)$, $a_2 = \frac{1}{9}(a-b-3c)$, $a_3 = \frac{1}{9}(\ldots)$

S.T the set 'S' = $\{(1,2,1), (2,1,0), (1,-1,2)\}$ form a basis of a vector space $V_3(F)$.

S.T a set 'S' = $\{(2,1,4)\ (1,-1,2)\ (3,1,-2)\}$ forms a basis of vector space $V_3(F)$ $a_1 = \frac{3b+c}{c}$ $a_2 = \frac{6a-16b+c}{24}$ $a_3 = \frac{2a-c}{8}$

1) S.T the set $S = \{(1,1,1)\ (-1,1,1)\ (1,0,1)\}$ forms a basis of R^3

2) S.T the set $S = \{(2,1,0)\ (2,1,1)\ (2,2,1)\}$ forms a basis of R^3

3) S.T the set $S = \{(1,1,1,1)\ (0,1,1,1)\ (0,0,1,1)\ (0,0,0,1)\}$ forms a basis of R^3 $a_1 = a$ $a_2 = b-a$ $a_3 = c-b$ $a_4 = d-c$

4) S.T the vectors $\{(1,1,2)\ (1,2,5)\ (5,3,4)\}$ of $R^3(R)$ donot form a basis of $R^3(R)$.

Sol: Given, $S = \{\alpha_1, \alpha_2, \alpha_3\}$

Where $\alpha_1 = (1,1,2)$
$\alpha_2 = (1,2,5)$
$\alpha_3 = (5,3,4)$

To prove 's' is a basis, We S.T

i)

let there exist scalars $a_1, a_2, a_3 \in R$ such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \bar{0}$$

$$\Rightarrow a_1(1,1,2) + a_2(1,2,5) + a_3(5,3,4) = (0,0,0)$$

$$\Rightarrow a_1 + a_2 + 5a_3 = 0 \rightarrow ①$$
$$a_1 + 2a_2 + 3a_3 = 0 \rightarrow ②$$
$$2a_1 + 5a_2 + 4a_3 = 0 \rightarrow ③$$

The above equ's can be written in matrix

$$\begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 3 \\ 2 & 5 & 4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{ccc} 0 & 3 & -6 \\ 0 & 3 & +6 \\ \hline & 0 & 0 \end{array} \qquad \begin{array}{ccc} 1 & 2 & 3 \\ 1 & 1 & 5 \\ \hline 0 & 1 & -2 \end{array} \qquad \begin{array}{ccc} 2 & 5 & 4 \\ -2 & -2 & -10 \\ \hline 0 & 3 & -6 \end{array}$$

$\therefore$ co-efficient Matrix $= \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 3 \\ 2 & 5 & 4 \end{bmatrix}$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & -2 \\ 0 & 3 & -6 \end{bmatrix} R_3 \to R_3 - 3R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

The above matrix is an echlon form.

$\therefore$ Rank of $A = \rho(A) =$ No.of Non-zero rows in the echlon form $= 2$

And No.of variable = 3

No of ~~Vabia~~ variable $\neq$ No.of Non-zero rows

$\therefore$ The given set 's' is not linearly independent

$\therefore$'s' is not a basis of $R^3(R)$.

NOTE

If 's' is any linearly independent in $V(F)$, then 's' can be extended to form the basis of $V(F)$.

Dimension of a vector space :

If $V(F)$ is a vector space over the field F. then the no.of elements presenting any basis of $V(F)$ is said to be "dimension of a vector space $V(F)$", and it is denoted by $\dim V(F)$

NOTE :

$\dim V_n(F) = n$

Theorem 20 :

Let $V(F)$ be a finite dimensional vector space then any two basis of 'V' has the same no.of elements.

Proof : Given $V(F)$ be a finite dimensional vector space over the field 'F'.

let $S_1 = \{\alpha_1, \alpha_2 \ldots \alpha_m\}$ and

$S_2 = \{\beta_1, \beta_2 \ldots \alpha_n\}$ are any two basis of $V(F)$.

Now, We show that $m = n$

Case(i): consider S_1 is the basis and S_2 is any linearly independent set $V(F)$.

If S_1 is the basis of $V(F)$.

$\Rightarrow \dim V(F) =$ The no.of vector present in $S_1 = m$

By one of the theorem, WKT the linearly independent set can be extended to form the basis of $V(F)$

$\therefore n \leq m \rightarrow (1)$

Case (ii): consider S_2 is the basis and S_1 is any linearly independent set $V(F)$.

If S_2 is the basis of $V(F)$

$\Rightarrow \dim V(F) =$ Th no.of vector present in $S_2 = n$

Again by one of the theorems S_1 can be extended to form the basis of $V(F)$.

$\therefore m \le n \quad \text{---} (2)$

$\therefore$ From (1) & (2) $\Rightarrow$ $n \le m, \ m \le n$

$\Rightarrow m = n$

Co-ordinates:

Let $V(F)$ be a finite dimensional vector space and $S = \{\alpha_1, \alpha_2 \dots \alpha_n\}$ be an order basis of V. If $\alpha \in V$ α can be uniquely expressed as,

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n, \text{ Where } a_1, a_2, \dots a_n \in F$$

The scalars $a_1, a_2 \dots a_n$ are called co-ordinats of α

The matrix $X = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ is called the co-ordinate matrix of 'α'.

Find the coordinates of the vectors $(2,1,-6)$ of R^3 Relative to the order basis $\{(1,1,2)\ (3,-1,0)\ (2,0,-1)\}$

Sol: Given $\alpha = (2,1,-6)$

let $S = \{\alpha_1, \alpha_2, \alpha_3\}$

Where $\alpha_1 = (1,1,2)$

$\alpha_2 = (3,-1,0)$

$\alpha_3 = (2,0,-1)$

Let there exist scalars $a_1, a_2, a_3 \in R$ such that

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$$

$\Rightarrow (2,1,-6) = a_1(1,1,2) + a_2(3,-1,0) + a_3(2,0,-1)$

$\Rightarrow (2,1,-6) = a_1 + a_2 + 2a_3, \ a_1 - a_2 + 0, \ 2a_1 + 0 - a_3$

$\therefore a_1 + a_2 + a_3 = 2$

$a_1 - a_2 = 1$

$2a_1 - a_3 = -6$

The above equ's can be written in matrix form as follows

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ co-efficient matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} R_4 \to R_4 - R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_2 \to R_2 - R_1$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The above matrix is an echlcon matrix

$\therefore$ Rank of $A = \rho(A) =$ no of non-zero rows in the echelon form 3 and no.of variables = 3

$\therefore$ No of variables = No of non-zero rows

ii) $L(S) = R^3$;

Let $(a,b,c) \in R^3$ any vector in (R^3)

let their exists scalers $a_1, a_2, a_3, a_4 \in R$ such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 = (a,b,c,d) \to (a)$$

$$\Rightarrow a_1(1,1,1,1) + a_2(0,1,1,1) + a_3(0,0,1,1) + a_4(0,0,0,1) = (a,b,c,d)$$

$a_1 = a \to (1)$

$a_1 + a_2 = b \to (2)$

$a_1 + a_2 + a_3 = c \to (3)$

$a_1 + a_2 + a_3 + a_4 = d \to (4)$

3) If W_1 and W_2 are the sub space of $V_4(R)$ generated by $\{(1,1,0,-1)\ (1,2,3,0)\ (2,3,3,1)\}$ and $\{(1,2,2,-2)\ (2,3,2,-3)\ (1,3,4,-3)\}$ respectively, find the dimensional of W_1, W_2, $W_1 \cap W_2$ and $W_1 + W_2$.

Sol: Given that $W_1 = \{(1,1,0,-1)\ (1,2,3,0)\ (2,3,3,-1)\}$.

$W_2 = \{(1,2,2,-2)\ (2,3,2,-3)\ (1,3,4,-3)\}$

To find ~~dimension~~ dim (W_1).

Arranging the vectors in the first set as rows of matrix and reducing to echlon form.

$$\Rightarrow W_1 = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & 1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\begin{array}{cccc} 1 & 2 & 3 & 0 \\ -1 & -1 & 0 & +1 \\ \hline 0 & 1 & 3 & 1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} R_3 \to R_3 - R_2$$

$$\begin{array}{cccc} 2 & 3 & 3 & -1 \\ -2 & -2 & 0 & +2 \\ \hline 0 & 1 & 3 & 1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here Non-zero rows are $\{(1,1,0,1)\ (0,1,3,1)\}$

$\Rightarrow \boxed{\dim(W_1) = 2}$

To Find $\dim(W_2)$:

Arranging the vectors in the second set as a rows of a matrix and reducing to echlon form.

$$\Rightarrow W_2 = \begin{bmatrix} 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\begin{array}{cccc} 2 & 3 & 2 & -3 \\ -2 & -4 & -4 & +4 \\ \hline 0 & -1 & -2 & 1 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix} R_3 \to R_3 + R_2$$

$$\begin{array}{cccc} 1 & 3 & 4 & -3 \\ -1 & -2 & -2 & +2 \\ \hline 0 & 1 & 2 & -1 \end{array}$$

3) sol: Given $s = \{\alpha_1, \alpha_2, \alpha_3\}$

Where $\alpha_1 = (2,1,4)$

$\alpha_2 = (1,-1,2)$

$\alpha_3 = (3,1,-2)$

To prove 's' is a basis, we show that

i) 's' is linearly independent:

let $\exists$ scalers $a_1, a_2, a_3 \in R$ such that $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \bar{0}$

$\Rightarrow a_1(2,1,4) + a_2(1,-1,2) + a_3(3,1,-2) = (0,0,0)$

$2a_1 + a_2 + 3a_3 = 0 \rightarrow ①$

$a_1 - a_2 + a_3 = 0 \rightarrow ②$

$4a_1 + 2a_2 - 2a_3 = 0 \rightarrow ③$

The above equ can be written in matrix form as follows

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ co-efficient matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix}$ $R_2 \rightarrow 2R_2 - R_1$, $R_3 \rightarrow R_3 - 2R_1$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -1 \\ 0 & 0 & -8 \end{bmatrix}$$

The above matrix is an echlon matrix.

$\therefore$ Rank of $A = \rho(A) =$ no of non-zero rows in the echlon from is 3 and no of variables $= 3$

$\therefore$ No. of variables $=$ no of non-zero rows in the echlon from is 3

$\therefore$ The given set 's' is linearly independent.

ii) $L(S) = V_3(F)$:

let $(a,b,c) \in V_3(F)$ be any vector in $V_3(F)$.

Let $\exists$ scalers $a_1, a_2, a_3 \in R$ such that $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = (a,b,c) \rightarrow ⑥$

$\Rightarrow a_1(2,1,4) + a_2(1,-1,2) + a_3(3,1,-2) = (a,b,c)$

$\Rightarrow 2a_1 + a_2 + 3a_3 = a$

$a_1 - a_2 + a_3 = b$

$4a_1 + 2a_2 - 2a_3 = c$

The above equ's can be matrix form as follows.

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_4 \to 2R_4 - R_3$

$R_5 \to R_5 - R_4$

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here Non-zero rows are $\{(1,1,0,-1)\ (0,1,3,1)\ (0,0,-2,-4)\}$

$\Rightarrow \boxed{\dim(W_1+W_2) = 3}$

To Find $\dim(W_1 \cap W_2)$:

We know that $\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

$\Rightarrow \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1+W_2)$

$= 2 + 2 - 3$

$= 4 - 3$

$\therefore \boxed{\dim(W_1 \cap W_2) = 1}$

$\dim(W_1 \cap W_2) =$

$\dim(W_1) = 2$,

$\dim(W_2) = 1$

$\dim(W_1+W_2) = 3$

4) If W_1 and W_2 are the sub space of $V_4(R)$ generated by $\{(1,1,-1,2)\ (2,1,3,0)\ (3,2,2,2)\}$ and $\{(1,-1,0,1)\ (-1,1,0,-1)\}$ respectively then find dim W_1, W_2, $W_1 \cap W_2$ and W_1+W_2

Imp

5) Let W_1 and W_2 be two sub spaces of R^4 given by $W_1 = \{(a,b,c,d) / b-2c+d=0\}$, $W_2 \{(a,b,c,d) / a=d, b=2c\}$ Find the basis and dimension of

i) W_1 (ii) W_2 (iii) $W_1 \cap W_2$ and hence Find $\dim(W_1+W_2)$

Sol: Given $W_1 = \{(a,b,c,d) / b-2c+d=0\}$

$W_2 = \{(a,b,c,d) / a=d, b=2c\}$ are any two subs of R^4.

put $a_3 = 1$ in equ (2) $\Rightarrow a_2 = -1 + 6$

$\Rightarrow a_2 = 5$

put a_2 in equ (1) $\Rightarrow a_1 = 15 - 3$

$\Rightarrow a_1 = 3 - 5$

$\Rightarrow a_1 = -2$

put a_1 in equ (1) $\Rightarrow -2 + a_4 = 2$

$\Rightarrow a_4 = 2 + 2$

$\Rightarrow a_4 = 4$

$\therefore$ The co-ordinate matrix of $(2, 3, 4, -1)$ are $\begin{bmatrix} -2 \\ 5 \\ 1 \\ 4 \end{bmatrix}$

1) If W is a sub space of $V_4(R)$ generated by the vectors $(1,-2,5,-3)$, $(2,3,1,-4)$ and $(3,8,-3,-5)$ then find a basis of W and its dimensional

Sol: Given $W = \{(1,-2,5,-3)\ (2,3,1,-4)\ (3,8,-3,-5)\}$

Aranging the given vectors as a rows of a matrix and redacing to echlon form

$\Rightarrow W = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix}$ $R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 3R_1$

$\begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix}$ $R_3 \rightarrow R_3 - 2R_2$

$\begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$3 + 6 - 15$

Here the Non-zero rows are $\{(1,-2,5,-3)\ (0,7,-9,2)\}$ is a basis of W

$\therefore \dim(W) = 2$

2) Find the basis of W and its dimention. $\dim(W) = 2$

a) $W = \{(1,2,0)\ (-1,0,1)\ (0,2,1)\}$ of $V_3(R)$

b) $W = \{(1,7,2)\ (1,-1,0)\ (1,2,1)\ (0,3,1)\}$ of R^3 $\dim(W) = 2$

$$\begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here the Non-zero rows are $\{(1,2,2,-2)\ (0,-1,-2,1)\}$

$\therefore \boxed{\dim(W_2) = 2}$

To Find $\dim(W_1+W_2)$.

Arranging the given vector in W_1 and W_2 as a rows of matrix and reducing to echelon form.

$$\Rightarrow W_1+W_2 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \\ 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix}$$

$R_2 \to R_2-R_1,\ R_3 \to R_3-2R_1$
$R_4 \to R_4-R_1,\ R_5 \to R_5-2R_1$
$R_6 \to R_6-R_1$

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{bmatrix}$$

$R_3 \to R_3-R_2,\ R_4\ R_4-R_2$
$R_5 \to R_5-R_2,\ R_6 \to R_6-2R_2$

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$

$R_3 \leftrightarrow R_6$

To Find (i):

If $(a,b,c,d) \in W_1 \Rightarrow b-2c+d=0$
$\Rightarrow b = 2c-d$

$\therefore (a,b,c,d) = (a, 2c-d, c, d)$
$= a(1,0,0,0) + c(0,2,1,0) + d(0,-1,0,1)$

$\therefore$ The set $\{(1,0,0,0)\ (0,2,1,0)\ (0,-1,0,1)\}$ is linearly indep and form the basis of W_1.

$\therefore \boxed{\dim(W_1) = 3}$

To Find (ii):

If $(a,b,c,d) \in W_2 \Rightarrow a=d,\ b=2c$

$(a,b,c,d) = (d, 2c, c, d)$
$= c(0,2,1,0) + d(1,0,0,1)$

$\therefore$ The set $\{(0,2,1,0)\ (1,0,0,1)\}$ is linearly independent form the basis of W_2.

$\therefore \boxed{\dim(W_2) = 2}$

To Find (iii).

If $(a,b,c,d) \in W_1 \cap W_2$
$\Rightarrow b-2c+d=0 \longrightarrow (1)$
and $a=d,\ b=2c$

put $b=2c$ in equ (1) $\Rightarrow \cancel{2c}-\cancel{2c}+d=0$
$\Rightarrow d=0$
$\Rightarrow a=0$
$\therefore (a,b,c,d) = (0,2c,c,0)$
$= c(0,2,1,0)$

$\therefore$ The set $\{(0,2,1,0)\}$ is linearly indepent and forms the ba of $W_1 \cap W_2$.

$\therefore \boxed{\dim(W_1 \cap W_2) = 1}$

We know that $\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim($
$= 3+2-1$
$= 4$

put $b_1 = 0, b_2 = 0, \ldots b_k = 0$ in equ (1), we get

$\Rightarrow c_1\gamma_1 + c_2\gamma_2 + \ldots + c_k\gamma_k + a_1\alpha_1 + a_2\alpha_2 + \ldots + a_m\alpha_m = 0$

since γ's and α's are the vectors present in the basis B_1 of W_1

∴ by definition of basis linearly independent

$\Rightarrow c_1 = c_2 = \ldots = c_k = a_1 = a_2 = \ldots = a_m = 0$

$c_1 = c_2 = \ldots = c_k = a_1 = a_2 = \ldots = a_m = b_1 = b_2 = \ldots = b_k = 0$

∴ The set S' is linearly independent.

To prove $L(S') = W_1 + W_2$:

We know that S' is a sub set of $W_1 + W_2$

∴ $L(S') \subseteq W_1 + W_2 \rightarrow (4)$

Let $\delta \in W_1 + W_2$

$\Rightarrow \delta = \alpha + \beta$, where $\alpha \in W_1, \beta \in W_2$ (∵ Definition of linear sum)

= (linear combination of α's and γ's) + (linear combination of β's and γ's)

$\Rightarrow \delta =$ (linear combination of α's, β's and γ's) $\in L(S')$

$\Rightarrow \delta \in L(S')$

∴ If $\delta \in W_1 + W_2 \Rightarrow \delta \in L(S')$

$\Rightarrow W_1 + W_2 \subseteq L(S') \rightarrow (5)$

∴ From (4) & (5) $\Rightarrow L(S') \subseteq W_1 + W_2,\ W_1 + W_2 \subseteq L(S')$

$\Rightarrow L(S') = W_1 + W_2$

∴ S' is a basis of $W_1 + W_2$

$\Rightarrow \dim(W_1 + W_2) = k + m + t$

∴ $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

Hence the proof

3) Sol: Given $\alpha = (2,3,4,-1)$

let $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$

Where $\alpha_1 = (1,1,1,2)$

$\alpha_2 = \cancel{(0,0,1,1)}\ (+1,-1,0,0)$

$\alpha_3 = (0,0,1,1)$

$\alpha_4 = (0,1,0,0)$

let there exist scalar $a_1, a_2, a_3, a_4 \in R$ such that

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4$$

$$(2,3,4,-1) = a_1(1,1,1,2) + a_2(1,-1,0,0) + a_3(0,0,1,1) + a_4(0,1,0,0)$$

$$(2,3,4,-1) = (a_1+a_2,\ a_1-a_2+a_4,\ a_1+a_3,\ 2a_1+a_3)$$

$\therefore$ $a_1 + a_2 = 2 \longrightarrow$ ①

$a_1 - a_2 + a_4 = 3 \longrightarrow$ ②

$a_1 + a_3 = 4 \longrightarrow$ ③

$2a_1 + a_3 = -1 \longrightarrow$ ④

solving ③ & ④

$$a_1 + \cancel{a_3} = 4$$
$$-2a_1 - a_3 = +1$$
$$-a_1 = 5 \Rightarrow \boxed{a_1 = -5}$$

put a_1 value in equ ① $\Rightarrow -5 + a_2 = 2$

$\Rightarrow a_2 = 2+5$

$\Rightarrow \boxed{a_2 = 7}$

put a_1, a_2 value in equ ② $\Rightarrow -5-7+a_4 = 3$

$\Rightarrow a_4 = 3+12$

$\Rightarrow \boxed{a_4 = 15}$

put a_1 value in equ ③ $\Rightarrow -5 + a_3 = 4$

$\Rightarrow \boxed{a_3 = 9}$

$\therefore$ The co-ordinate matrix of $(2,3,4,-1)$ are $\begin{bmatrix} -5 \\ 7 \\ 9 \\ 15 \end{bmatrix}$

4)
i) Sol: Given $\alpha = (1,0,-1)$

let $S = \{\alpha_1, \alpha_2, \alpha_3\}$

Where $\alpha_1 = (0,1,-1)$

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$$

$$(4,5,6) = a_1(1,1,1) + a_2(-1,1,1) + a_3(1,0,-1)$$

$$(4,5,6) = (a_1 - a_2 + a_3,\ a_1 + a_2,\ a_1 + a_2 - a_3)$$

$$\therefore\ a_1 - a_2 + a_3 = 4 \rightarrow (1)$$

$$a_1 + a_2 = 5 \rightarrow (2)$$

$$a_1 + a_2 - a_3 = 6 \rightarrow (3)$$

solving (1) & (3)

$$a_1 - \cancel{a_2} + \cancel{a_3} = 4$$

$$a_1 + \cancel{a_2} - \cancel{a_3} = 6$$

$$2a_1 = 10$$

$$a_1 = 5$$

put a_1 in equ (2) $\rightarrow 5 + a_2 = 5$

$$\Rightarrow a_2 = 0$$

put a_1, a_2 in equ (3) $\Rightarrow 5 + 0 - a_3 = 6$

$\cancel{\Rightarrow 10 - a_3 = 6}$ $\Rightarrow a_3 = 6 - 5$

$\cancel{\Rightarrow -a_3 = -10 + 6}$ $\Rightarrow a_3 = -1$

$\cancel{\Rightarrow a_3 = +4}$

$\therefore$ The co-ordinate matrix of $(4,5,6)$ are $\begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}$

Sol: Given $\alpha = (2,3,4,-1)$

Let $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$

Where $\alpha_1 = (1,1,0,0)$

$\alpha_2 = (0,1,1,0)$

$\alpha_3 = (0,0,1,1)$

$\alpha_4 = (1,0,0,0)$.

let there exist scalars $a_1, a_2, a_3, a_4 \in R$ such that

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4$$

$$(2,3,4,-1) = a_1(1,1,0,0) + a_2(0,1,1,0) + a_3(0,0,1,1) + a_4(1,0,0,0)$$

$$(2,3,4,-1) = (a_1 + a_4,\ a_1 + a_2,\ a_2 + a_3,\ a_3)$$

$$\therefore\ a_1 + a_4 = 2 \rightarrow (1)$$

$$a_1 + a_2 = 3 \rightarrow (2)$$

$$a_2 + a_3 = 4 \rightarrow (3)$$

$$a_3 = -1 \rightarrow (4)$$

$\alpha_2 = (1,1,0)$

$\alpha_3 = (1,0,2)$

let there exist scalars $a_1, a_2, a_3 \in R$ such that

$\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$

$(1,0,-1) = a_1(0,1,-1) + a_2(1,1,0) + a_3(1,0,2)$

$(1,0,-1) = (a_2+a_3,\ a_1+a_2,\ -a_1+2a_3)$

$\therefore\ a_2 + a_3 = 1 \rightarrow (1)$

$a_1 + a_2 = 0 \rightarrow (2)$

$-a_1 + 2a_3 = -1 \rightarrow (3)$

Equ (2) & (3) $\Rightarrow$

solving (2) & (3)

$a_1 + a_2 = 0$

$-a_1 + 2a_3 = -1$

$a_2 + 2a_3 = -1 \rightarrow (4)$

solving (1) & (4)

$a_2 + a_3 = 1$

$a_2 + 2a_3 = -1$

$-a_3 = 2$

$a_3 = 2$

put a_3 in equ (1) $\Rightarrow a_2 - 2 = 1$

$a_2 = 3$

put a_2 in equ (2) $\Rightarrow a_1 + 3 = 0$

$\Rightarrow a_1 = -3$

$\therefore$ The co-ordinate matrix of $(1,0,-1)$ are $\begin{bmatrix} -3 \\ 3 \\ -2 \end{bmatrix}$

ii) sol Given $\alpha = (4,5,6)$

let $S = \{\alpha_1, \alpha_2, \alpha_3\}$

where $\alpha_1 = (1,1,1)$

$\alpha_2 = (-1,1,1)$

$\alpha_3 = (1,0,-1)$

Let there exist scalars $a_1, a_2, a_3 \in R$ such that

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & 9 \end{bmatrix}$$

The above matrix is an ~~echd~~ echelon form

Rank of $A = \rho(A) =$ No. of non-zero rows in the echelon form $= 3$

And No. of variables $= 3$

No. of variables $=$ No. of non-zero rows

$\therefore$ The given set 'S' is linearly independent

ii) $L(S) = V_3(F)$:

Let $a, b, c \in V_3(F)$ be any vector in $V_3(F)$

Let there exist scalars $a_1, a_2, a_3 \in F$ such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = (a, b, c) \longrightarrow (a)$$

$$a_1(1,2,1) + a_2(2,1,0) + a_3(1,1,2) = (a,b,c)$$

$$\therefore a_1 + 2a_2 + a_3 = a$$

$$2a_1 + a_2 \; a_3 = b$$

$$a_1 + 2a_3 = c$$

The above equ's can be written in matrix form as follows

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\therefore \text{Augmented matrix} = \left[\begin{array}{ccc:c} 1 & 2 & 1 & a \\ 2 & 1 & -1 & b \\ 1 & 0 & 2 & c \end{array}\right] \quad R_2 \to R_2 - 2R_1, \; R_3 \to R_3 - R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & a \\ 0 & 3 & -3 & b-2a \\ 0 & -2 & 1 & c-a \end{array}\right] \quad R_3 \to 3R_3 - 2R_2$$

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & a \\ 0 & 3 & -3 & b-2a \\ 0 & 0 & 9 & a-2b+3c \end{array}\right]$$

The above matrix can be written in equ's form as follows.

$\Rightarrow (2,1) = (a_1+a_2, a_2)$

$\therefore a_1 + a_2 = 2 \rightarrow ①$

$a_2 = 1 \rightarrow ②$

$a_1 = -5$
$a_2 = 1$
$a_3 = 9$
$a_4 = 15$

put $a_2 = 1$ in equ ① $\Rightarrow a_1 + 1 = 2$

$\Rightarrow a_1 = 1$

$\therefore$ The co-ordinate matrix of $(2,1)$ are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

3) Find the co-ordinate matrix of $(2,3,4,-1)$ with respect to the basis $B = \{(1,1,1,2), (1,-1,0,0), (0,0,1,1), (0,1,0,0)\}$ of $V_4(R)$

4) Find the co-ordinates of

i) $(1,0,-1)$ relative to the basis $\{(0,1,-1)\ (1,1,0)\ (1,0,2)\}$ ($a_1 = 3$, $a_2 = 3$, $a_3 = -2$)

ii) $(4,5,6)$ with respect to the order basis $\{(1,1,1)\ (1,1,1)\ (1,0,\ldots$ ($a_1 = 5$, $a_2 = 0$, 0)

iii) $(2,3,4,-1)$ with respect to the order basis $\{(1,1,0,0)\ (0,1,1,0)\ (0,0,1,\ldots$ $(1,0,0,0)\}$. ($a_1 = 2$, $a_3 = 5$, $a_4 = 4$)

Dimension of a sub space :

imp

Theorem 21 :

let W_1 and W_2 the two sub space of a finite dimensional vector space $V(F)$. then. ~~dimen~~ , $\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim\ldots$

sol: proof : Given W_1 and W_2 are any two sub spaces of $V(F)$

$\Rightarrow W_1 \cap W_2$ is also a sub space of $V(F)$.

since, $V(F)$ is a finite dimensional

$\therefore W_1 \cap W_2$ is also finite dimensional

Let $S = \{\gamma_1, \gamma_2 \ldots \gamma_k\}$ be a basis of $W_1 \cap W_2$

$\Rightarrow \dim(W_1 \cap W_2) = k$

If 'S' is a basis of $W_1 \cap W_2$

$\Rightarrow$ 'S' is L.I and $L(S) = W_1 \cap W_2$

$\therefore$ 'S' is L.I in W_1

$\Rightarrow$ 'S' can be extended to form the basis of W_1

let $B_1 = \{\gamma_1, \gamma_2 \ldots \gamma_k, \alpha_1, \alpha_2 \ldots \alpha_m\}$ be a basis of W_1

$\Rightarrow \dim(W_1) = k+m$

$\therefore$ 'S' is also a linearly independent in S can be extended to form the basis of W_2.

put a_1 value in equ ⑤ $\to a_2 = b - a$

put a_1, a_2 value in Equ ⑥ $\to a_1 + a_2 + a_3 = c$

$$a + b - a + a_3 = c$$

$$a_3 = c - b$$

put a_1, a_2, a_3 values in equ ⑦ $= a_1 + a_2 + a_3 + a_4 = d$

$$\Rightarrow a + b - a + c - d + a_4 = d$$

$$\Rightarrow a_4 = d - c$$

put a_1, a_2, a_3, a_4 value in equ ⓐ

$\Rightarrow (a, b, c, d) = a(1,1,1,1) + b - a(0,1,1,1) + c - b(0,0,1,1) + d - c(0,0,0,1)$

$\therefore$'s' is linearly independent and $L(S) = V$

$\therefore$'s' is basis of $V_3(R)$.

2) sol: Given $S = \{\alpha_1, \alpha_2, \alpha_3\}$

where $\alpha_1 = (1,2,1)$

$\alpha_2 = (2,1,0)$

$\alpha_3 = (1,-1,2)$

To prove 's' is an basis we show that

i) 's' is Linearly independent:

let there exist scalars $a_1, a_2, a_3 \in F$ such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \bar{0}$$

$$\Rightarrow a_1(1,2,1) + a_2(2,1,0) + a_3(1,-1,2) = (0,0,0)$$

$$\therefore a_1 + 2a_2 + a_3 = 0 \to ①$$

$$2a_1 + a_2 - a_3 = 0 \to ②$$

$$a_1 + 2a_3 = 0 \to ③$$

The above equ's can be written in matrix form as follows:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ co-efficient matrix $= \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -3 \\ 0 & -2 & 1 \end{bmatrix}$$ $R_3 \to 3R_3 - 2R_2$

$$\Rightarrow 2a_1 + 2a_2 + 2a_3 = a \rightarrow (4)$$

$$a_2 + a_3 = c \rightarrow (5)$$

$$2a_3 = 2b - a$$

$$a_3 = \frac{2b-a}{2}$$

put a_3 value can be written in equ (5) $\Rightarrow a_2 = c - \left(\frac{2b-a}{2}\right)$

$$a_2 = \frac{2c-2b+a}{2}$$

put a_2, a_3 values in equ (4) $\rightarrow 2a_1 + 2a_2 + 2a_3 = a$

$$\Rightarrow 2a_1 + \cancel{2}\left[\frac{2c-2b+a}{\cancel{2}}\right] + \cancel{2}\left[\frac{2b-a}{\cancel{2}}\right] = a$$

$$\Rightarrow 2a_1 + 2c - \cancel{2b} + \cancel{a} + \cancel{2b} - \cancel{a} = a$$

$$\Rightarrow 2a_1 = a - 2c$$

$$\Rightarrow a_1 = \frac{a-2c}{2}$$

put a_1, a_2, a_3 value in equ (a)

$$\rightarrow (a_1, a_2, a_3) = \frac{1}{2}(a-2c)(2,1,0) + \frac{1}{2}(2c-2b+a)(-1,1,1) +$$

$$\frac{1}{2}\left(\frac{2b-a}{2}\right)(2,2,1)$$

∴ 's' is linearly independent and $L(S) = V$

∴ 's' is basis of R^3

5) sol: Given $S = \{\alpha_1, \alpha_2, \alpha_3\}$

$$\alpha_1 = (1,1,1,1)$$

$$\alpha_2 = (0,1,1,1)$$

$$\alpha_3 = (0,0,1,1)$$

$$\alpha_4 = (0,0,0,1)$$

To prove 's' is a basis, We show that

1) 's' is linearly independent:

let their exist scalars $a_1, a_2, a_3 \in R$ such that $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$

$$\rightarrow a_1(1,1,1,1) + a_2(0,1,1,1) + a_3(0,0,1,1) + a_4(0,0,0,1) = (0,0,0,0)$$

$$\Rightarrow a_1 + 0a_2 + 0a_3 + 0a_4 = 0 \rightarrow (1)$$

$$a_1 = 0 \rightarrow (1)$$

$$\Rightarrow a_1 + a_2 = 0 \rightarrow (2)$$

$$\Rightarrow a_1 + a_2 + a_3 = 0 \rightarrow (3)$$

$$\Rightarrow a_1 + a_2 + a_3 + a_4 = 0 \rightarrow (4)$$

The above equ can be written matrix form as follows.

$$\begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 2 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

The above matrix is an echlon matrix

$\therefore$ Rank of $A = \rho(A) =$ no.of non-zero rows in the echlon

$= 3$ and no of variables $= 3$

No.of variables = No.of non-zero rows

$\therefore$ The given set 'S' is linearly independent.

(ii) $L(S) = R^3$:

Let $(a,b,c) \in R^3$ be any vector in R^3

let their exist scalers $a_1, a_2, a_3 \in R$ such that $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 =$ (a,b,c) $\rightarrow$ (a)

$\Rightarrow a_1(2,1,0) + a_2(2,1,1) + a_3(2,2,1) = (a,b,c)$

$$\Rightarrow 2a_1 + 2a_2 + 2a_3 = a$$
$$a_1 + a_2 + 2a_3 = b$$
$$a_2 + a_3 = c$$

The above equation can be matrix form as follows

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\therefore$ Augumented matrix $= \begin{bmatrix} 2 & 2 & 2 & : a \\ 1 & 1 & 2 & : b \\ 0 & 1 & 1 & : c \end{bmatrix}$ $R_2 \rightarrow 2R_2 - R_1$

$$\begin{bmatrix} 2 & 2 & 2 & : a \\ 0 & 0 & 2 & : 2b-a \\ 0 & 1 & 1 & : c \end{bmatrix} R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 2 & 2 & 2 & : a \\ 0 & 1 & 1 & : c \\ 0 & 0 & 2 & : 2b-a \end{bmatrix}$$

The above matrix can be written in Equ's as follow

$\Rightarrow 2a_2 = b - a + b - c$

$\Rightarrow a_2 = \frac{2b - a - c}{2}$

put a_2, a_3 values in equ (iii)

$\Rightarrow a_1 - a_2 + a_3 = a$

$\Rightarrow a_1 - \left(\frac{2b - a - c}{2}\right) + (b - c) = a$

$\Rightarrow a_1 = a + \left(\frac{2b - a - c}{2}\right) - (b - c)$

$\Rightarrow a_1 = 2a + 2b - a - c - 2b + 2c$

$\Rightarrow a_1 = a + c$

put a_1, a_2, a_3 values in equ (a)

$\Rightarrow (a,b,c) = \frac{1}{2}(a+c)(1,1,1) + \frac{1}{2}(2b - a - c)(-1,1,1) + (b - c)(1,0,-1)$

$\therefore$'s' is linearly independent and $L(S) = V$

$\therefore$'s' is basis of R^3.

5) sol: Given $S = \{\alpha_1, \alpha_2, \alpha_3\}$

where $\alpha_1 = (2,1,0)$

$\alpha_2 = (2,1,1)$

$\alpha_3 = (2,2,1)$

To prove 's' is a basis, we show that

1) 's' is linearly independent:

let there exist scalers $a_1, a_2, a_3 \in R$ such that

$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \bar{0}$

$\Rightarrow a_1(2,1,0) + a_2(2,1,1) + a_3(2,2,1) = (0,0,0)$

$\Rightarrow 2a_1 + 2a_2 + 2a_3 = 0 \rightarrow ①$

$a_1 + a_2 + 2a_3 = 0 \rightarrow ②$

$0 + a_2 + a_3 = 0 \rightarrow ③$

The above equ's can be written matrix form as follows.

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The ~~at~~ $\therefore$ Co-efficient matrix $= \begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ $R_2 \rightarrow 2R_2 - R_1$

No. of variables = No. of Non zero rows

$\therefore$ The given set 'S' is linearly independent.

ii) $L(S) = R^3$

Let $(a,b,c) \in R^3$ be any vector in R^3

let their exist scalers $a_1, a_2, a_3 \in R$ such that

$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = (a,b,c) \rightarrow$ (a)

$\Rightarrow a_1(1,1,1) + a_2(-1,1,1) + a_3(1,0,-1) = (0,0,0)$

$\Rightarrow a_1 - a_2 + a_3 = 0 \rightarrow (1)$

$a_1 + a_2 + 0 = 0 \rightarrow (2)$

$a_1 + a_2 - a_3 = 0 \rightarrow (3)$

The above equ's can be written in matrix form as follows

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\therefore$ Augmented matrix = $\begin{bmatrix} 1 & 1 & 1 & : a \\ 1 & 1 & 0 & : b \\ 1 & 1 & 3 & : c \end{bmatrix}$ $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & -1 & 1 & : a \\ 0 & 2 & -1 & : b-a \\ 0 & 2 & -2 & : c-a \end{bmatrix} R_3 \rightarrow R_3 - R_2$$

$$\begin{bmatrix} 1 & -1 & 1 & : a \\ 0 & 2 & -1 & : b-a \\ 0 & 0 & -1 & : c-b \end{bmatrix}$$

$\therefore$ The above matrix can be written in equation as follows

$a_1 - a_2 + a_3 = a \rightarrow (4)$

$2a_2 - a_3 = b - a \rightarrow (5)$

$-a_3 = c - b$

$a_3 = b - c$

put a_3 values in Equ (5) $\Rightarrow 2a_2 - (b-c) = b - a$

$\Rightarrow 2a_2 - b + c = b - a$

$\Rightarrow (a,b,c) = \frac{1}{6}\left[\frac{2b+c}{6}\right](2,1,4) + \frac{-1}{24}(-16b+6a+c)(1,-1,2) + \frac{1}{8}$
$(3,1,2$

$\therefore$ s' is linear independent and $L(S) = V$

$\therefore$'s' is basis of $V_3(F)$

Sol: Given $S = \{\alpha_1, \alpha_2, \alpha_3\}$

Where $\alpha_1 = (1,1,1)$

$\alpha_2 = (-1,1,1)$

$\alpha_3 = (1,0,-1)$

To prove 's' is a basis, We show that

i) 's' is linearly independent:

let $\exists$ scalers $a_1, a_2, a_3 \in R$ such that $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \bar{0}$

$\Rightarrow a_1(1,1,1) + a_2(-1,1,1) + a_3(1,0,-1) = (0,0,0)$

$\Rightarrow a_1 - a_2 + a_3 = 0 \rightarrow (1)$

$a_1 + a_2 + 0 = 0 \rightarrow (2)$

$a_1 + a_2 - a_3 = 0 \rightarrow (3)$

The above equs can be written matrix form as follow

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ co-efficient matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$ $R_2 \rightarrow R_2 - R_1$

$R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 2 & -2 \end{bmatrix}$$ $R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

The above matrix is an echlon matrix

$\therefore$ Rank of $A = \rho(A)$ = no.of Non-zero rows in the echlo form is 3 and no.of variables = 3.

$a_1 + 2a_2 + a_3 = a \rightarrow (4)$

$0a_1 - 3a_2 - 3a_3 = b - 2a \rightarrow (5)$

$0a_1 + 0a_2 + 9a_3 = a - 2b + 3c$

$\Rightarrow 9a_3 = a - 2b + 3c$

$\Rightarrow a_3 = \frac{1}{9}(a - 2b + 3c)$

put a_3 value in equ (5) $\Rightarrow -3a_2 - \cancel{3}\left(\frac{a - 2b + 3c}{\cancel{9}_3}\right) = b - 2a$

$\Rightarrow -9a_2 - a + 2b - 3c = 3b - 6a$

$\Rightarrow -9a_2 = -5a + b + 3c$

$\Rightarrow a_2 = \frac{-(-5a + b + 3c)}{9}$

$\Rightarrow a_2 = \frac{1}{9}(5a - b - 3c)$

put a_2, a_3 values in equ (4)

$\Rightarrow a_1 + 2\left(\frac{5a - b - 3c}{9}\right) + \frac{a - 2b + 3c}{9} = a$

$\Rightarrow 9a_1 + 10a - 2b - 6c + a - 2b + 3c = 9a$

$\Rightarrow 9a_1 = (-2a + 4b + 3c)$

$\Rightarrow a_1 = \frac{1}{9}(-2a + 4b + 3c)$

put a_1, a_2, a_3 values in equ (a)

$\Rightarrow (a,b,c) = \frac{1}{9}(-2a + 4b + 3c)(1,2,1) + \frac{1}{9}(5a - b - 3c)(2,1,0) +$

$\frac{1}{9}(a - 2b + 3c)(1,1,2)$

$\therefore$ 'S' is linearly independent and $L(S) = V$

$\therefore$ 'S' is a basis of $V_3(F)$.

co-ordinates

Find the co-ordinate matrix of the vector $(2,1)$ of $V_2(R)$ the order basis $\{(1,0)\ (1,1)\}$.

sol: Given $\alpha = (2,1)$

let $S = \{\alpha_1, \alpha_2\}$

here $\alpha_1 = (1,0)$

$\alpha_2 = (1,1)$

let there exist scalars $a_1, a_2 \in R$ such that

$\alpha = a_1\alpha_1 + a_2\alpha_2$

$(2,1) = a_1(1,0) + a_2(1,1)$

let $B_2 = \{\gamma_1, \gamma_2, \ldots \gamma_k, \beta_1, \beta_2 \ldots$

$\Rightarrow \dim(W_2) = k+t$

$\therefore \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = k+m+\cancel{k}+t-\cancel{k}$
$= k+m+t$

Now, We show that $\dim(W_1+W_2) = k+m+t$:

for this, We can show that is a basis containg 'k+m+t' vectors

let $S' = \{\gamma_1, \gamma_2 \ldots \gamma_k, \alpha_1, \alpha_2 \ldots \alpha_m\}, \beta_1, \beta_2 \ldots \beta_t\}$ is any non-Empty sub set of W_1+W_2.

To prove that S' is linearly independent

let ∃ scalars $c_1, c_2 \ldots c_k, a_1, a_2 \ldots a_m, b_1, b_2 \ldots b_t \in F$

such that $c_1\gamma_1 + c_2\gamma_2 + \ldots + c_k\gamma_k + a_1\alpha_1 + a_2\alpha_2 + \ldots + a_m\gamma_m + b_1\beta_1 + b_2\beta_2 + \ldots + b_t\beta_t = \bar{0} \rightarrow (1)$

$\Rightarrow b_1\beta_1 + b_2\beta_2 + \ldots + b_t\beta_t = -c_1\gamma_1 - c_2\gamma_2 \ldots - c_k\gamma_k - a_1\alpha_1 - a_2\alpha_2 - \ldots - a_m\alpha_m$

$= (-c_1)(\gamma_1) + (-c_2)\gamma_2 + \ldots + (-c_k)\gamma_k + (-a_1)\alpha_1 + (-a_2)\alpha_2 + \ldots + (-a_m)\alpha_m$

$\Rightarrow b_1\beta_1 + b_2\beta_2 + \ldots + b_t\beta_t \in W_1 \rightarrow (2)$

We can Writte, $b_1\beta_1 + b_2\beta_2 + \ldots + b_t\beta_t = 0\gamma_1 + 0\gamma_2 + \ldots + 0\gamma_k + b_1\beta_1 + b_2\beta_2 + \ldots + b_t\beta_t$

$\Rightarrow b_1\beta_1 + b_2\beta_2 + \ldots + b_t\beta_t \in W_2 \rightarrow (3)$

From (2) & (3)

$\Rightarrow b_1\beta_1 + b_2\beta_2 + \ldots + b_t\beta_t \in W_1 \cap W_2$

$\Rightarrow$ their exist scalars $d_1, d_2 \ldots d_k \in F$ such that

$b_1\beta_1 + b_2\beta_2 + \ldots + b_t\beta_t = d_1\gamma_1 + d_2\gamma_2 + \ldots + d_k\gamma_k$ [since S is basis of $W_1 \cap W_2$]

$\Rightarrow b_1\beta_1 + b_2\beta_2 + \ldots + b_t\beta_t + (-d_1)\gamma_1 + (-d_2)\gamma_2 + \ldots + (-d_k)\gamma_k = \bar{0}$

$\Rightarrow b_1 = b_2 = \ldots b_t = d_1 = d_2 = \ldots d_k = 0$

since, β's and γ's are the vectors in the basis B_2 W_2.

$\therefore$ By definition of basis B_2 is linearly Independent.

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\therefore$ Augmented matrix $\left[\begin{array}{ccc:c} 2 & 1 & 3 & a \\ 1 & -1 & 1 & b \\ 4 & 2 & -2 & c \end{array}\right]$ $R_2 \to 2R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$$\left[\begin{array}{ccc:c} 2 & 1 & 3 & a \\ 0 & -3 & -1 & 2b-a \\ 0 & 0 & -8 & c-2a \end{array}\right]$$

$\therefore$ The above matrix can be written in equations as follows

$\Rightarrow 2a_1 + a_2 + 3a_3 = a \to (4)$

$\Rightarrow -3a_2 - a_3 = 2b - a \to (5)$

$\Rightarrow -8a_3 = c - 2a$

$\Rightarrow a_3 = \frac{c-2a}{-8} = \frac{2a-c}{8}$

put a_3 value in equ (5) $\Rightarrow -3a_2 - a_3 = 2b - a$

$\Rightarrow -3a_2 - \left(\frac{2a-c}{8}\right) = 2b - a$

$\Rightarrow -3a_2 = 2b - a + \left[\frac{2a-c}{8}\right]$

$\Rightarrow -3a_2 = \frac{16b - 8a + 2a - c}{8}$

$\Rightarrow a_2 = \frac{-16b + 8a - 2a + c}{24}$

$\Rightarrow a_2 = \frac{-16b + 6a + c}{24}$

put a_2, a_3 value in equ (4)

$\Rightarrow 2a_1 + a_2 + 3a_3 = a$

$\Rightarrow 2a_1 + \left(\frac{-16b + 6a + c}{24}\right) + 3\left(\frac{2a-c}{8}\right) = a$

$\Rightarrow 2a_1 = a - \left(\frac{6a - 3c}{2}\right) - \left[\frac{16b + 6a + c}{24}\right]$

$\Rightarrow 2a_1 = \frac{24a - 3(6a - 3c) + 16b - 6a - c}{24}$

$\Rightarrow a_1 = \frac{\cancel{24a} - \cancel{18a} + 9c + 16b - \cancel{6a} - c}{24}$

$= \frac{1}{2}\left[\frac{8c + 16b}{24}\right]$

$= \frac{1}{\cancel{2}} \cdot \left[\frac{\cancel{8}^{4}(c + 2b)}{\cancel{24}\, 6}\right] \Rightarrow a_1 = \frac{c + 2b}{6}$

$\boxed{\therefore \dim(W_1+W_2) = 4}$

Let W_1 and W_2 be two sub space ~~of~~ of R^4 given by $W_1 = \{(a,b,c,d) / b+c+d=0\}$ and $W_2 = \{(a,b,c,d) / \cancel{a=d, b=2c}$ $a+b=0, c=2d\}$

Find the basis and dimension of i) W_1 ii) W_2 iii) $W_1 \cap W_2$ and hence find $\dim(W_1+W_2)$

$\dim(W_1) = 3$
$\dim(W_2) = 2$
$\dim(W_1 \cap W_2) =$
$\dim(W_1+W_2) =$

Sol: Given $W_1 = \{(a,b,c,d) / b+c+d=0\}$
$W_2 = \{(a,b,c,d) / a+b=0, c=2d\}$

To Find (i):

If $a,b,c,d \in W_1 \Rightarrow b+c+d=0$
$\Rightarrow b = -c-d$

$\therefore (a,b,c,d) = (a,-c-d,c,d)$
$\Rightarrow a(1,0,0,0) + c(0,-1,1,0) + d(0,-1,0,1)$

$\therefore$ The set $\{(1,0,0,0)\ (0,-1,1,0)\ (0,-1,0,1)\}$ is L.I and forms as basis of W_1.

$\therefore \dim(W_1) = 3$

To Find (ii):

If $a,b,c,d \in W_2 \Rightarrow a+b=0, c=2d$
$\Rightarrow a=-b, c=2d$

$\therefore (a,b,c,d) = (-b,b,2d,d)$
$= b(-1,1,0,0) + d(0,0,2,1)$

$\therefore$ The set $\{(-1,1,0,0)\ (0,0,2,1)\}$ is L.I and forms as basis of W_2.

$\therefore \dim(W_2) = 2$

To Find (iii):

If $(a,b,c,d) \in W_1 \cap W_2$
$\Rightarrow b+c+d=0 \rightarrow (1)$
and $a=-b, c=2d$
put $c=2d$ in equ (1) $\Rightarrow b+2d+d=0$
$\Rightarrow b+3d=0$
$\Rightarrow b=-3d$

$\therefore a=+3d$

$\therefore (a,b,c,d) = (3d,-3d,2d,d)$

$= \alpha(3,-2,2,1)$

$\therefore$ The set $\{(3,-2,2,1)\}$ is L.I and form a basis of $W_1 \cap W_2$

$\therefore \dim(W_1 + W_2) = 3+2-1$

$= 4$

$\therefore \dim(W_1 + W_2) = 4$

1) Sol: Given $W = \{(1,2,0)\ (-1,0,1)\ (0,2,1)\}$

Arrange the given vectors as a rows of a matrix and reducing to echelon form

$\Rightarrow W = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} R_2 \to R_2 + R_1$

$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} R_3 \to R_3 - R_2$

$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ Here the non-zero rows are $\{(1,2,0)\ (-1,0,1)\ (0,2,1)\}$ a basis of $V_3(R)$.

$\Rightarrow \dim(W) = 2$

2) Sol: Given $W = \{(2,7,3)\ (1,-1,0)\ (1,2,1)\ (0,3,1)\}$

Arrange the given vectors as a rows of a matrix and reducing to echelon-form.

$\Rightarrow W = \begin{bmatrix} 2 & 7 & 3 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix} R_1 \to R_1 - R_2$

$\sim \begin{bmatrix} 1 & 8 & 3 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & 8 & 3 \\ 0 & -9 & -3 \\ 0 & -6 & -2 \\ 0 & 3 & 1 \end{bmatrix} R_3 \to R_3 + 3R_4$

$$\sim \begin{bmatrix} 1 & 8 & 3 \\ 0 & -9 & 3 \\ 0 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix} \begin{matrix} R_3 \to 3R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 8 & 3 \\ 0 & -9 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here the Non-zero rows are $\{(1,8,3)\ (0,-9,3)\}$ is a basis of R^3.

$\Rightarrow \dim(W) = 2$

Sol: Given that $W_1 = \{(1,1,-1,2)\ (2,1,3,0)\ (3,2,2,2)\}$

$W_2 = \{(1,-1,0,1)\ (-1,1,0,-1)\}$

$\dim(W_1)$:

Arranging the vectors in the first set has rows of a matrix and reducing to echelon form

$$\Rightarrow W_1 = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 1 & 3 & 0 \\ 3 & 2 & 2 & 2 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 5 & -4 \\ 0 & -1 & 5 & -4 \end{bmatrix} R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 5 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here Non-zero rows are $\{(1,1,-1,2),\ (0,-1,5,-4)\}$

$\Rightarrow \dim(W_1) = 2$

To find $\dim(W_2)$:

Arranging the vectors in the second set has rows of a matrix and reducing to echelon form

$$\Rightarrow W_2 = \begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 1 & 0 & -1 \end{bmatrix} R_2 \to R_2 + R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Here non-zero rows are $\{(1,-1,0,1)\}$

$\Rightarrow \dim(W_2) = 1$

To find $\dim(W_1+W_2)$

Arranging the vectors in W_1 & W_2 as a rows of a matrix & reducing to echelon form

$$\Rightarrow W_1+W_2 = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 1 & 3 & 0 \\ 3 & 2 & 2 & 2 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1,\ R_5 \to R_5 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 5 & -4 \\ 0 & -1 & 5 & -4 \\ 0 & -2 & 1 & -1 \\ 0 & 2 & -1 & 1 \end{bmatrix} \begin{matrix} R_3 \to R_3 - R_2 \\ R_5 \to R_5 + R_4 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 5 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 5 & -4 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here non-zero rows are $\{(1,1,-1,2)\ (0,-1,5,-4)\ (0,-2,1,-1)\}$

$\Rightarrow \dim(W_1+W_2) = 3$

To find $\dim(W_1 \cap W_2)$:

W.K.T, $\dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1+W_2)$

$= 2+1-3$

$= 3-3$

$= 0$

$\therefore \dim(W_1 \cap W_2) = 0$

7) Let 'V' is a vector space of polynomials over 'R' let W_1 & W_2 are the subspaces generated by $\{x^3+x^2-1, x^3+2x^2+3x, 2x^3+3x^2+x-1\}$ and $\{x^3+2x^2+2x-2, 2x^3+3x^2+2x-3, x^3+3x^2+4x-3\}$ respectively find i) $\dim(W_1)$ ii) $\dim(W_2)$ iii) $\dim(W_1+W_2)$ iv) $\dim(W_1 \cap$

sol: Given $W_1 = \{x^3+x^2-1, x^3+2x^2+3x, 2x^3+3x^2+x-1\}$

$W_2 = \{x^3+2x^2+2x-2, 2x^3+3x^2+2x-3, x^3+3x^2+4x-3\}$

The co-ordinates of the polynomials w.r.to basis $\{x^3, x^2, x, 1\}$ are

$W_1 = \{(1,1,0,-1)\ (1,2,3,0)\ (2,3,1,-1)\}$

$W_2 = \{(1,2,2,-2)\ (2,3,2,-3)\ (1,3,4,-3)\}$

To Find $\dim(W_1)$:

Arranging the vectors in the first set has rows of a matrix and reducing to echelon form.

$$\Rightarrow W_1 = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 1 & -1 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -2 & 0 \end{bmatrix} \quad C_3 \leftrightarrow C_4$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Here Non-zero rows are $\{(1,1,-1,0)\ (0,1,1,3)\ (0,0,0,-2)\}$

$\Rightarrow \dim(W_1) = 3$

To Find $\dim(W_2)$:

Arranging the vectors in the 2nd set has rows of a matrix and reducing to echelon form.

$$\Rightarrow W_2 = \begin{bmatrix} 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -1 & 2 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix} R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here Non-zero rows are $\{(1,2,2,-2)\ (0,-1,-2,1)\}$

$\Rightarrow \dim(W_2) = 2$

To Find $\dim(W_1 + W_2)$:

- Arranging the vectors in W_1 & W_2 as a rows of a matrix and reducing to echelon form

$$\Rightarrow (W_1 + W_2) = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 1 & -1 \\ 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1,\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1,\ R_5 \to R_5 - 2R_1 \\ R_6 \to R_6 \to R_6 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{bmatrix} \begin{matrix} R_5 \to R_5 - R_4 \\ R_6 \to R_6 - 2R_4 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here Non zero rows $\{(1,1,0,-1)\ (0,1,3,1)\ (0,1,1,1)\ (0,1,2,-1)\}$

$\Rightarrow \dim(W_1 + W_2) = 4$

To Find $\dim(W_1 \cap W_2)$.

WKT, $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

$\Rightarrow \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 + W_2)$

$= 3 + 2 - 4$

$\therefore \dim(W_1 \cap W_2) = 1$

8) If 'V' is space generated by the polynomials $\alpha = x^3 + 2x^2 - 2x + 1$, $\beta = x^3 + 3x^2 - x + 4$, $\gamma = 2x^3 + x^2 - 7x - 7$. Find a basis of 'V' and its dimension.

Sol: Given $\alpha = x^3 + 2x^2 - 2x + 1$

$\beta = x^3 + 3x^2 - x + 4$

$\gamma = 2x^3 + x^2 - 7x - 7$

Let $V = \{x^3 + 2x^2 - 2x + 1,\ x^3 + 3x^2 - x + 4,\ 2x^3 + x^2 - 7x - 7\}$

The co-ordinates of the polynomials w.r.to the basis $\{x^3, x^2, x, 1\}$

$\Rightarrow V = \{(1,2,-2,1)\ (1,3,-1,4)\ (2,1,-7,-7)\}$

Arranging the given vectors as a rows of a matrix and reducing to echelon-form.

$$\Rightarrow V = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 1 & 3 & -1 & 4 \\ 2 & 1 & -7 & -7 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & -3 & -3 & -9 \end{bmatrix} \quad R_3 \to R_3 + 3R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here Non-zero rows are $\{(1,2,-2,1)\ (0,1,1,3)\}$.

$\therefore$ The set $V = \{(1,2,-2,1)\ (0,1,1,3)\}$ is Linearly independent and forms the basis of 'V'.

$\therefore \dim(V) = 2$

quotient space:

If 'W' is a sub space of $V(F)$, then quotient space is denoted by $\frac{V}{W}$, define by

$$\frac{V}{W} = \{W+\alpha \mid \alpha \in V\}$$ is also a vector space

1) Let 'W' be a sub space of $V(F)$, then show that $\frac{V}{W}$ is a vector space over 'F', For the vector addition and scalar multiplication are defined by

i) $(W+\alpha)+(W+\beta) = W+(\alpha+\beta)$

ii) $a(W+\alpha) = W+a\alpha, \ \forall \alpha,\beta \in V(F), a \in F$

Sol: Given 'W' be a sub space of $V(F)$

Given set $\frac{V}{W} = \{W+\alpha \mid \alpha \in V\}$

given compositions are

i) $(W+\alpha)+(W+\beta) = W+(\alpha+\beta)$

ii) $a(W+\alpha) = W+a\alpha, \ \forall \alpha,\beta \in V(F), a \in F$

i) To prove $(\frac{V}{W}, +)$ is a commutative group:

Closure axiom: $\forall \alpha,\beta \in V \to \alpha+\beta \in V$

If $\alpha,\beta \in V \Rightarrow W+\alpha, W+\beta \in \frac{V}{W}$

then $(W+\alpha)+(W+\beta) = W+(\alpha+\beta) \in \frac{V}{W}$

$\therefore \frac{V}{W}$ is closed under the given composition addition.

Associative axiom:

$\forall \alpha,\beta,\gamma \in V \Rightarrow \alpha+(\beta+\gamma) = (\alpha+\beta)+\gamma$

If $\alpha,\beta,\gamma \in V \Rightarrow (W+\alpha), W+\beta, W+\gamma \in \frac{V}{W}$

consider $(W+\alpha)+[(W+\beta)+(W+\gamma)] = (W+\alpha)+[W+(\beta+\gamma)]$

$$= W+[\alpha+(\beta+\gamma)]$$

$$= W+[(\alpha+\beta)+\gamma]$$

$$= [W+(\alpha+\beta)]+(W+\gamma)$$

$$= [(W+\alpha)+(W+\beta)]+(W+\gamma)$$

$\therefore$ composition addition is associative in V/W.

$\forall\ a, b \in F$ and $W+\alpha, W+\beta \in \frac{V}{W}$ then

a) $a[(W+\alpha)+(W+\beta)] = a[W+(\alpha+\beta)]$

$= W + a(\alpha+\beta)$

$= W + (a\alpha + b\beta)$

$= (W+a\alpha) + (W+b\beta)$

$\therefore a[(W+\alpha)+(W+\beta)] = a(W+\alpha) + b(W+\beta)$

b) $(a+b)(W+\alpha) = W + (a+b)\alpha$

$= W + (a\alpha + b\alpha)$

$= (W+a\alpha) + (W+b\alpha)$

$\therefore (a+b)(W+\alpha) = a(W+\alpha) + b(W+\alpha)$

c) $(ab)(W+\alpha) = W + (ab)\alpha$

$= W + a(b\alpha)$

$= a(W+b\alpha)$

$\therefore (ab)(W+\alpha) = a[b(W+\alpha)]$

d) $1\cdot(W+\alpha) = W + 1\cdot\alpha$

$\therefore 1(W+\alpha) = W+\alpha$

$\therefore \frac{V}{W}$ is a vector space over the field 'F'.

NOTE :

1) The zero element of $\frac{V}{W}$ is $0+W = W$

2) $\alpha + W = \beta + W \iff \alpha - \beta \in W$.

3) $\alpha + W = W \iff \alpha \in W$.

Dimension of quotient space :

Imp v.imp **Theorem 22 :**

Let 'W' be a sub space of a finite dimensional vector space of $V(F)$ then $\dim\left(\frac{V}{W}\right) = \dim V - \dim W$

Proof : Given 'W' is a sub space of $V(F)$.

Where $V(F)$ is Finite dimensional

Let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be a basis of W

$\Rightarrow \dim(W) = m$

Identity axiom

$\forall\ \alpha \in V \Rightarrow \exists$ a vector $\bar{0} \in V$ such that

$$\alpha + \bar{0} = \bar{0} + \alpha = \alpha$$

If $\alpha, \bar{0} \in V \Rightarrow W+\alpha,\ W+\bar{0} \in \frac{V}{W}$

consider, $(W+\alpha) + (W+\bar{0}) = W + (\alpha + \bar{0})$

$= W + \alpha$

Similarly $(W+\bar{0}) + (W+\alpha) = W+\alpha$.

$\therefore\ W+\bar{0} = W$ is the identity element in $\frac{V}{W}$

Inverse axiom:

$\forall\ \alpha \in V \Rightarrow \exists$ a vector $-\alpha \in V$ such that

$$\alpha + (-\alpha) = (-\alpha) + \alpha = \bar{0}$$

If $\alpha, -\alpha \in V \Rightarrow W+\alpha,\ W+(-\alpha) \in \frac{V}{W}$

consider, $(W+\alpha) + [W+(-\alpha)] = W + [\alpha + (-\alpha)]$

$= W + [\alpha - \alpha]$

$= W + 0$

$= W$

Similarly $(W+(-\alpha)) + (W+\alpha) = W$

$\therefore$ Inverse axiom is satisfied.

Commutative axiom:

$\forall\ \alpha, \beta \in V \Rightarrow \alpha+\beta = \beta+\alpha$

Consider $(W+\alpha) + (W+\beta) = W + (\alpha+\beta)$

$= W + (\beta+\alpha)$

$= (W+\beta) + (W+\alpha)$

$(W+\alpha) + (W+\beta) = (W+\beta) + (W+\alpha)$

$\therefore$ commutative axiom is satisfied.

$\therefore\ (\frac{V}{W}, +)$ is a commutative group.

ii) Given condition (ii) satisfies external composition.

ie; scalar multiplication of a vector in $\frac{V}{W}$

$\therefore\ \forall\ a \in F,\ W+\alpha \in \frac{V}{W} \Rightarrow a(W+\alpha) = W + a\alpha \in \frac{V}{W}$

iii) To prove remaining condition:

If 'B' is a basis of 'W'

$\Rightarrow$ B is L.I and $L(B) = W$

Since, 'W' is a sub space of $V(F)$

$\therefore$ B' is also linearly independent set in $V(F)$

$\Rightarrow$ 'B' can be extended to form the basis of $V(F)$

Let $S = \{\alpha_1, \alpha_2, \dots \alpha_m, \beta_1, \beta_2, \dots \beta_l\}$ be a basis of $V(F)$

$\Rightarrow$ $\dim(V) = m + l$

$\therefore$ $\dim(V) - \dim(W) = (m+l) - m$

$= l$

$\therefore$ $\dim(V) - \dim(W) = l$

Next We show that $\dim\left(\frac{V}{W}\right) = l$:

i.e. To show that any sub set in $\left(\frac{V}{m}\right)$ containing 'l' elements forms a basis of $\left(\frac{V}{W}\right)$.

2) Show that the set $S=\{(1,2,1),(2,1,0),(1,-1,2)\}$ forms a basis of a vector space $V_3(F)$

3) Show that a set $S=\{(2,1,4),(1,-1,2),(3,1,-2)\}$ forms a basis of a vector space $V_3(F)$.

4) Show that the set $S=\{(1,1,1),(-1,1,1);(1,0,-1)\}$ forms a basis of R^3

5) Show that the set $S=\{(2,1,0),(2,1,1),(2,2,1)\}$ forms a basis of R^3.

6) Show that the set $S=\{(1,1,1,1),(0,1,1,1),(0,0,1,1),(0,0,0,1)\}$ forms a basis of R^3

7) Show that the vectors $S=\{(1,1,2),(1,2,5),(5,3,4)\}$ of $R^3(R)$ do not form a basis of $R^3(R)$

Q5
Sol) Given $S=\{\alpha_1,\alpha_2,\alpha_3\}$

where $\alpha_1=(2,1,0)$

$\alpha_2=(2,1,1)$

$\alpha_3=(2,2,1)$

To prove that 'S' is a basis, we show that

i) 'S' is linearly independent :-

Let there exist scalars $a_1,a_2,a_3 \in R$ such that

$$a_1\alpha_1+a_2\alpha_2+a_3\alpha_3=\bar{0}$$

$$\Rightarrow a_1(2,1,0)+a_2(2,1,1)+a_3(2,2,1)=(0,0,0)$$

$$\Rightarrow 2a_1+2a_2+2a_3=0 \rightarrow (1)$$

$$a_1+a_2+a_3=0 \rightarrow (2)$$

$$0+a_2+a_3=0 \rightarrow (3)$$

The above equ's can be written matrix form as follows

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ Co-efficient matrix $= \begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ $R_2 \to 2R_2 - R_1$

$$\begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 2 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

The above matrix is an echlon matrix

$\therefore$ Rank of $A = \rho(A) =$ no. of non-zero rows in the echlon form

$= 3$ and no. of variables $= 3$

$\therefore$ No. of variables $=$ No. of non zero rows

$\therefore$ The given set S' is linearly independent.

ii) $\underline{L(S) = R^3}$:-

Let $(a,b,c) \in R^3$ be any vector in R^3

Let their exist scalars $a_1, a_2, a_3 \in R$ such that

$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \overline{0}\ (a,b,c) \to (a)$

$\Rightarrow a_1(2,1,0) + a_2(2,1,1) + a_3(2,2,1) = (a,b,c)$

$\Rightarrow 2a_1 + 2a_2 + 2a_3 = a$

$a_1 + a_2 + a_3 = b$

$0 + a_2 + a_3 = c$

The above equation can be matrix form as follows

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\therefore$ Augmented matrix $= \begin{bmatrix} 2 & 2 & 2 & : & a \\ 1 & 1 & 2 & : & b \\ 0 & 1 & 1 & : & c \end{bmatrix}$ $R_2 \to 2R_2 - R_1$

$$\begin{bmatrix} 2 & 2 & 2 & : & a \\ 0 & 0 & 2 & : & 2b-a \\ 0 & 1 & 1 & : & c \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 2 & 2 & 2 & : & a \\ 0 & 1 & 1 & : & c \\ 0 & 0 & 2 & : & 2b-a \end{bmatrix}$$

$\therefore$ The above matrix can be written in eq's as follows

$\Rightarrow 2a_1 + 2a_2 + 2a_3 = a \to (4)$

$a_2 + a_3 = c \to (5)$

$2a_3 = 2b - a$

$a_3 = \frac{2b-a}{2}$

put a_3 value in eq (5) $\Rightarrow a_2 = c - \left(\frac{2b-a}{2}\right)$

$$a_2 = \frac{2c-2b+a}{2}$$

put a_2, a_3 values in eq (4) $\Rightarrow 2a_1 + 2a_2 + 2a_3 = a$

$$\Rightarrow 2a_1 + \cancel{2}\left[\frac{2c-2b+a}{\cancel{2}}\right] + \cancel{2}\left[\frac{2b-a}{\cancel{2}}\right]$$

$$\Rightarrow 2a_1 + 2c - \cancel{2b} + \cancel{a} + \cancel{2b} - \cancel{a} = a$$

$$\Rightarrow 2a_1 = a - 2c$$

$$\Rightarrow a_1 = \frac{a-2c}{2}$$

put a_1, a_2, a_3 values in eq(a)

$$\Rightarrow (a_1, a_2, a_3) = \frac{1}{2}(a-2c)(2,1,0) + \frac{1}{2}(2c-2b+a)(-1,1,1) + \frac{1}{2}\left(\frac{2b-a}{2}\right)(2,2,1)$$

$\therefore$ 'S' is linearly independent and $L(S) = V$

$\therefore$ 'S' is basis of $\mathbb{R}^3$

6 sol) Given $S = \{\alpha_1, \alpha_2, \alpha_3\}$

$$\alpha_1 = (1,1,1,1)$$
$$\alpha_2 = (0,1,1,1)$$
$$\alpha_3 = (0,0,1,1)$$
$$\alpha_4 = (0,0,0,1)$$

To prove 'S' is a basis, we show that

i) 'S' is linearly independent :

Let their exist scalars $a_1, a_2, a_3 \in R$ such that $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \bar{0}$

$\Rightarrow a_1(1,1,1,1) + a_2(0,1,1,1) + a_3(0,0,1,1) + a_4(0,0,0,1) = (0,0,0,0)$

$\Rightarrow a_1 + 0a_2 + 0a_3 + 0a_4 = 0 \rightarrow (1)$

$a_1 = 0 \rightarrow (1)$

$\Rightarrow a_1 + a_2 = 0 \rightarrow (2)$

$\Rightarrow a_1 + a_2 + a_3 = 0 \rightarrow (3)$

$\Rightarrow a_1 + a_2 + a_3 + a_4 = 0 \rightarrow (4)$

The above equ can be written matrix as follows.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ Co-efficient matrix $= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ $R_4 \rightarrow R_4 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_3 \rightarrow R_3 - R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_2 \rightarrow R_2 - R_1$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The above matrix is an echlon matrix

$\therefore$ Rank of $A = \rho(A)$ = no of non-zero rows in the echlon form 3 a_4

no. of variables = 3

$\therefore$ No. of variables = No of non-zero rows

ii) $\underline{L(S) = R^3}$:-

Let $(a, b, c) \in R^3$ any vector in (R^3)

Let their exists scalars $a_1, a_2, a_3, a_4 \in R$ such that

$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 = (a, b, c, d) \rightarrow (a)$

$a_1(1,1,1,1) + a_2(0,1,1,1) + a_3(0,0,1,1) + a_4(0,0,0,1) = (a, b, c, d)$

$a_1 = a \rightarrow (1)$

$a_1 + a_2 = b \rightarrow (2)$

$a_1 + a_2 + a_3 = c \rightarrow (3)$

$a_1 + a_2 + a_3 + a_4 = d \rightarrow (4)$

put a_1 value in eq(5) $\Rightarrow a_2 = b - a$

put a_1, a_2 value in eq(6) $\Rightarrow a_1 + a_2 + a_3 = c$

$a + b - a + a_3 = c$

$a_3 = c - b$

put a_1, a_2, a_3 values in eq (7) $\pm$ $a_1 + a_2 + a_3 + a_4 = d$

$\Rightarrow a + b - a + c - d + a_4 = d$

$\Rightarrow a_4 = d - c$

put a_1, a_2, a_3, a_4 value in eq (a)

$\Rightarrow (a, b, c, d) = a(1,1,1,1) + b - a(0,1,1,1) + c - d(0,0,1,1) + d - c(0,0,0$

$\therefore$ 'S' is linearly independent and $L(S) = V$

$\therefore$ 'S' is basis of $V_3(R)$.

3 sol) Given $S = \{\alpha_1, \alpha_2, \alpha_3\}$

where $\alpha_1 = (2, 1, 4)$

$\alpha_2 = (1, -1, 2)$

$\alpha_3 = (3, 1, 2)$

To prove 'S' is a basis, we show that

i) 'S' is linearly independent :-

Let $\exists$ scalars $a_1, a_2, a_3 \in R$ such that $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \bar{0}$

$\Rightarrow a_1(2,1,4) + a_2(1,-1,2) + a_3(3,1,-2) = (0,0,0)$

$2a_1 + a_2 + 3a_3 = 0 \rightarrow (1)$

$a_1 - a_2 + a_3 = 0 \rightarrow (2)$

$4a_1 + 2a_2 - 2a_3 = 0 \rightarrow (3)$

The above eq can be written in matrix form as follows

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ Co-efficient matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix}$

$R_2 \rightarrow 2R_2 - R_1$

$R_3 \rightarrow R_3 - 2R_1$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -1 \\ 0 & 0 & -8 \end{bmatrix}$$

The above matrix is an echlon matrix

$\therefore$ Rank of $A = \rho(A) =$ no of non-zero rows in the echlon form is 3 and no of variables $= 3$

$\therefore$ No. of variables = no of non zero rows in the echlon form

$\therefore$ The given set 'S' is linearly independent.

ii) $L(S) = V_3(F)$

Let $(a, b, c) \in V_3(F)$ be any vector in $V_3(F)$

Let $\exists$ scalars $a_1, a_2, a_3 \in R$ such that $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = (a,$

$\Rightarrow a_1(2,1,4) + a_2(1,-1,2) + a_3(3,1,-2) = (a, b, c)$

$\Rightarrow 2a_1 + a_2 + 3a_3 = a$

$\Rightarrow a_1 - a_2 + a_3 = b$

$4a_1 + 2a_2 - 2a_3 = c$

The above eq can be matrix form as follows

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\therefore$ Augumented matrix $\begin{bmatrix} 2 & 1 & 3 & : & a \\ 1 & -1 & 1 & : & b \\ 4 & 2 & -2 & : & c \end{bmatrix}$

$R_2 \to 2R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 2 & 1 & 3 & : & a \\ 0 & -3 & -1 & : & 2b-a \\ 0 & 0 & -8 & : & c-2a \end{bmatrix}$$

$\therefore$ The above matrix can be written in equations as follows

$\Rightarrow 2a_1 + a_2 + 3a_3 = a \rightarrow (4)$

$\Rightarrow -3a_2 - a_3 = 2b - a \rightarrow (5)$

$\Rightarrow -8a_3 = c - 2a$

$\Rightarrow a_3 = \frac{c-2a}{-8} = \frac{2a-c}{8}$

put a_3 value in eq (5) $\Rightarrow -3a_2 - a_3 = 2b - a$

$\Rightarrow -3a_2 - \left(\frac{2a-c}{8}\right) = 2b - a$

$\Rightarrow -3a_2 = 2b - a + \left[\frac{2a-c}{8}\right]$

$\Rightarrow -3a_2 = \frac{16b - 8a + 2a - c}{8}$

$a_2 = \frac{-16b + 8a - 2a + c}{24}$

$\Rightarrow a_2 = \frac{-16b + 6a + c}{24}$

put a_2, a_3 values in eq (a)

$$\Rightarrow 2a_1 + a_2 + 3a_3 = a$$

$$\Rightarrow 2a_1 + \left[\frac{-16b+6a+c}{24}\right] + 3\left(\frac{2a-c}{8}\right) = a$$

$$\Rightarrow 2a_1 = a - \left[\frac{6a-3c}{2}\right] - \left[\frac{16b+6a+c}{24}\right]$$

$$\Rightarrow 2a_1 = \frac{24a - 3(6a-3c) + 16b - 6a - c}{24}$$

$$a_1 = \frac{\cancel{24a} - \cancel{18a} + 9c + 16b - \cancel{6a} - c}{24}$$

$$= \frac{1}{2}\left[\frac{8c+16b}{24}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{8}(c+2b)}{\cancel{24}_6}\right] \Rightarrow a_1 = \frac{c+2b}{6}$$

put a_1, a_2, a_3 value in eq (a)

$$\Rightarrow (a,b,c) = \frac{1}{6}\left[\frac{2b+c}{6}\right](2,1,4) + \frac{1}{24}(-16b+6a+c)(1,1,2) + \frac{1}{8}(3,1,-3)$$

$\therefore$ S' is linear independent and $L(S) = V$

$\therefore$ 'S' is basis of $V_3(F)$

4 sol:- Given $S = \{\alpha_1, \alpha_2, \alpha_3\}$

where, $\alpha_1 = (1,1,1)$

$\alpha_2 = (-1,1,1)$

$\alpha_3 = (1,0,-1)$

To prove 'S' is a basis, we show that

i) 'S' is linearly independent:

let $\exists$ scalars $a_1, a_2, a_3 \in R$ such that $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \bar{0}$

$$\Rightarrow a_1(1,1,1) + a_2(-1,1,1) + a_3(1,0,-1) = (0,0,0)$$

$\Rightarrow a_1 - a_2 + a_3 = 0 \rightarrow (1)$

$a_1 + a_2 + 0 = 0 \rightarrow (2)$

$a_1 + a_2 - a_3 = 0 \rightarrow (3)$

The above equs can be written matrix form as follows

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ co-efficient matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$ $R_2 \rightarrow R_2 - R_1$

$R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 2 & -2 \end{bmatrix} R_3 \rightarrow R_3 - R_2$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

The above matrix is an echlon matrix

$\therefore$ Rank of $A = \rho(A) =$ no. of non zero rows in the echlon form is 3 and no. of variables $= 3$.

$\therefore$ No. of variables $=$ No. of non-zero rows

$\therefore$ The given set 'S' is linearly independent

ii) $L(S) = R^3$:

Let $(a,b,c) \in R^3$ be any vector in R^3

Let their exist scalars $a_1, a_2, a_3 \in R$ such that

$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = (a,b,c) \rightarrow (a)$

$\Rightarrow a_1(1,1,1) + a_2(-1,1,1) + a_3(1,0,-1) = (0,0,0)$

$\Rightarrow a_1 - a_2 + a_3 = 0 \rightarrow (1)$

$\Rightarrow a_1 + a_2 + 0 = 0 \rightarrow (2)$

$a_1 + a_2 - a_3 = 0 \rightarrow (3)$

Dimension of a sub space :-

Theorem-21 :

Let W_1 and W_2 be two sub spaces of a finite dimensional vector space $V(F)$ then $\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

Proof :-

Given W_1 and W_2 are any two sub spaces of $V(F)$.

$\Rightarrow W_1 \cap W_2$ is also a sub space of $V(F)$.

Since $V(F)$ is finite dimensional

$\therefore W_1 \cap W_2$ is also finite dimensional.

Let $S = \{\gamma_1, \gamma_2, \dots \gamma_k\}$ be a basis of $W_1 \cap W_2$

$\Rightarrow \dim(W_1 \cap W_2) = k$

If 'S' is a basis of $W_1 \cap W_2$

$\Rightarrow$ 'S' is linearly independent and $L(S) = W_1 \cap W_2$

$\therefore$ 'S' is linearly independent set in W_1

$\Rightarrow$ 'S' can be extended to form the basis of W_1

Let $B_1 = \{\gamma_1, \gamma_2, \dots \gamma_k, \alpha_1, \alpha_2, \dots \alpha_m\}$ be a basis of W_1

$\Rightarrow \dim(W_1) = k+m$

$\therefore$ 'S' is also a linearly independent set in W_2

$\Rightarrow$ 'S' can be ~~en~~ extended to form the basis of W_2

Let $B_2 = \{\gamma_1, \gamma_2, \dots \gamma_k, \beta_1, \beta_2, \dots \beta_t\}$ be a basis of W_2

$\Rightarrow \dim(W_2) = k+t$

$\therefore \dim(W_1) \cancel{+W_2)} + \dim(W_1) - \dim(W_1 \cap W_2) = k+m+k+t-k$

$= k+m+t$

Now, <u>we show that $\dim(W_1+W_2) = k+m+t$:</u>

For these, we can show that $\exists$ a basis containing "$k+m+t$" vectors

Let $S' = \{\gamma_1, \gamma_2, \dots \gamma_k, \alpha_1, \alpha_2, \dots \alpha_m, \beta_1, \beta_2, \dots \beta_t\}$. is any non-empty sub set of W_1+W_2.

<u>To prove that S' is linearly independent :-</u>

Let $\exists$ scalars $c_1, c_2, \dots c_k, a_1, a_2, \dots a_m, b_1, b_2, \dots b_t \in$

Such that

$c_1\gamma_1 + c_2\gamma_2 + \cdots + c_k\gamma_k + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m + b_1\beta_1 + b_2\beta_2 + \cdots + b_t\beta_t = \bar{0} \rightarrow (1)$

$\Rightarrow b_1\beta_1 + b_2\beta_2 + \cdots + b_t\beta_t = -c_1\gamma_1 - c_2\gamma_2 \cdots c_k\gamma_k - a_1\alpha_1 - a_2\alpha_2 \cdots a_m\alpha_m$

$= (-c_1)\gamma_1 + (-c_2)\gamma_2 + \cdots + (-c_k)\gamma_k + (-a_1)\alpha_1 + (-a_2)\alpha_2 + \cdots + (-a_m)\alpha_m$

$\Rightarrow b_1\beta_1 + b_2\beta_2 + \cdots + b_t\beta_t \in W_1 \rightarrow (2)$.

We can write,

$b_1\beta_1 + b_2\beta_2 + \cdots + b_t\beta_t = 0\gamma_1 + 0\gamma_2 + \cdots + 0\gamma_k + b_1\beta_1 + b_2\beta_2 + \cdots + b_t\beta_t$

$\Rightarrow b_1\beta_1 + b_2\beta_2 + \cdots + b_t\beta_t \in W_2 \rightarrow (3)$

$\therefore$ from eqs (2) and (3)

$\Rightarrow b_1\beta_1 + b_2\beta_2 + \cdots + b_t\beta_t \in W_1 \cap W_2$

$\Rightarrow \exists$ scalars $d_1, d_2 \cdots d_k \in F$ such that

$b_1\beta_1 + b_2\beta_2 + \cdots + b_t\beta_t = d_1\gamma_1 + d_2\gamma_2 + \cdots + d_k\gamma_k$ ($\because S$ is a basis of $W_1 \cap W_2$)

$\Rightarrow b_1\beta_1 + b_2\beta_2 + \cdots + b_t\beta_t + (-d_1)\gamma_1 + (-d_2)\gamma_2 + \cdots + (-d_k)\gamma_k = \bar{0}$

$\Rightarrow b_1 = b_2 = \cdots = b_t = d_1 = d_2 = \cdots = d_k = 0$

since β's and γ's are the vectors in the basis B_2 of W_2

$\therefore$ By definition of basis of B_2 is linearly independent

put $b_1 = 0, b_2 = 0, \cdots b_t = 0$ in eq(1), we get

$\Rightarrow c_1\gamma_1 + c_2\gamma_2 + \cdots c_k\gamma_k + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m = 0$

Since γ's and α's are the vectors, present in the basis B_1

$\therefore$ By definition of basis, B_1 is linearly independent

$\Rightarrow c_1 = c_2 = \cdots = c_k = a_1 = a_2 = \cdots = a_m = 0$.

$\therefore c_1 = c_2 = \cdots = c_k = a_1 = a_2 = \cdots a_m = b_1 = b_2 = \cdots b_t = 0$

$\therefore$ The set S' is linearly independent

<u>To prove $L(S') = W_1 + W_2$:-</u>

$\Rightarrow (a,b,c) = \frac{1}{2}(a+c)(1,1,1) + \frac{1}{2}(2b-a-c)(-1,1,1) + (b-c)(1,\ldots$

$\therefore$'s' is linearly independent and $L(s) = V$

$\therefore$ s' is basis of R^3.

2) Find the co-ordinate matrix of the vector $(2,1)$ of $V_2(\ldots)$ in the ordered basis $\{(1,0),(1,1)\}$

Sol:- Given $\alpha = (2,1)$

Let $S = \{\alpha_1, \alpha_2\}$

where $\alpha_1 = (1,0)$

$\alpha_2 = (1,1)$

Let $\exists$ scalars $a_1, a_2 \in R$ such that.

$\alpha = a_1\alpha_1 + a_2\alpha_2$

$\Rightarrow (2,1) = a_1(1,0) + a_2(1,1)$

$\Rightarrow (2,1) = (a_1 + a_2, a_2)$

$\therefore a_1 + a_2 = 2 \longrightarrow (1)$

$a_2 = 1 \rightarrow (2)$

put a_2 value in eq (1)

$a_1 + 1 = 2$.

$a_1 = 2 - 1$

$\boxed{a_1 = 1}$

$\therefore$ The coordinate matrix of $(2,1)$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The above eq's can be written in matrix form as follows

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\therefore$ Augmented matrix $= \left[\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 1 & 1 & 0 & b \\ 1 & 1 & -3 & c \end{array}\right]$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 0 & 2 & -1 & b-a \\ 0 & 0 & -1 & c-b \end{array}\right]$$

$\therefore$ The above matrix can be written in equation as follows

$$a_1 - a_2 + a_3 = a \rightarrow (4)$$
$$2a_2 - a_3 = b - a \rightarrow (5)$$
$$-a_3 = c - b$$
$$a_3 = b - c$$

put a_3 values in eq (5) $\Rightarrow 2a_2 - (b-c) = b-a$

$\Rightarrow 2a_2 - b + c = b - a$

$\Rightarrow 2a_2 = b - a + b - c$

$\Rightarrow a_2 = \frac{2b-a-c}{2}$

put a_3, a_2 values in eq (4)

$\Rightarrow a_1 - a_2 + a_3 = a$

$\Rightarrow a_1 - \left[\frac{2b-a-c}{2}\right] + (b-c) = a$

$\Rightarrow a_1 = a + \left[\frac{2b-a-c}{2}\right] - (b-c)$

$\Rightarrow a_1 = 2a + 2b - a - c - 2b + 2c$

$\Rightarrow a_1 = a + c$

put a_1, a_2, a_3 values in eq (a)

We know that 'S'' is a subset of $W_1 + W_2$

$\Rightarrow L(S') \subseteq W_1 + W_2 \rightarrow (4)$

Let $\delta \in W_1 + W_2$

$\Rightarrow \delta = \alpha + \beta$, where $\alpha \in W_1$ & $\beta \in W_2$ ($\because$ definition of linear sum)

$=$ (linear combination of α's and γ's) + (L.C of β's and γ's)

$\Rightarrow \delta =$ (linear combination of α's, β's, and γ's) $\in L(S')$

$\Rightarrow \delta \in L(S')$

$\therefore$ If $\delta \in W_1 + W_2 \Rightarrow \delta \in L(S')$

$\Rightarrow W_1 + W_2 \subseteq L(S') \rightarrow (5)$

$\therefore$ From (4) and (5) $\Rightarrow L(S') \subseteq W_1 + W_2$, $W_1 + W_2 \subseteq L(S')$

$\Rightarrow L(S') = W_1 + W_2$

$\therefore S'$ is a basis of $W_1 + W_2$

$\Rightarrow \dim(W_1 + W_2) = k + m + t$

$\because \dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

$\therefore$ Hence the proof.

3) Find the co-ordinates matrix of $(2,3,4,-1)$ with respect to the basis $B=\{(1,1,1,2),(1,-1,0,0),(0,0,1,1),(0,1,0,0)\}$ of $V_4(R)$

Sol: Given $\alpha=(2,3,4,-1)$

Let $S=\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$

where $\alpha_1=(1,1,1,2)$

$\alpha_2=(1,-1,0,0)$

$\alpha_3=(0,0,1,1)$

$\alpha_4=(0,1,0,0)$

Let $\exists$ scalar $a_1,a_2,a_3,a_4 \in R$ such that

$$\alpha = a_1\alpha_1+a_2\alpha_2+a_3\alpha_3+a_4\alpha_4$$

$$(2,3,4,-1)=a_1(1,1,1,2)+a_2(1,-1,0,0)+a_3(0,0,1,1)+a_4(0,1,0,0)$$

$$(2,3,4,-1)=(a_1+a_2,\ a_1-a_2+a_4,\ a_1+a_3,\ 2a_1+a_3)$$

$\therefore$ $a_1+a_2=2 \rightarrow (1)$

$a_1-a_2+a_4=3 \rightarrow (2)$

$a_1+a_3=4 \rightarrow (3)$

$2a_1+a_3=-1 \rightarrow (4)$

Solving (3) and (4)

$$a_1+a_3=4$$
$$-2a_1-a_3=1$$

$$-a_1=5 \Rightarrow \boxed{a_1=-5}$$

put a_1 value in eq (1) $\Rightarrow -5+a_2=2$

$\Rightarrow a_2=2+5$

$a_2=7$

put a_1,a_2 value in eq (2) $\Rightarrow -5-7+a_4=3$

$\Rightarrow a_4=3+12$

$\Rightarrow \boxed{a_4=15}$

put a_1 value in eq (3) $\Rightarrow -5+a_3=4$

$\Rightarrow a_3=9$

$\therefore$ The co-ordinate matrix of $(2,3,4,-1)$ are $\begin{bmatrix}-5\\7\\9\\15\end{bmatrix}$

4) Find the coordinate of

i) $(1,0,-1)$ relative to the basis $\{(0,1,-1),(1,1,0),(1,0,2)\}$

ii) $(4,5,6)$ with r.t the order basis $\{(1,1,1),(-1,1,1),(1,0,-1)\}$

iii) $(2,3,4,-1)$ with r.t the order basis $\{(1,1,0,0),(0,1,1,0)(0,0,1,1)$
$(1,0,0,0)\}$

Sol: Given $\alpha=(1,0,-1)$

Let $S=\{\alpha_1,\alpha_2,\alpha_3\}$

where $\alpha_1=(0,1,-1)$

$\alpha_2=(1,1,0)$

$\alpha_3=(1,0,2)$

Let $\exists$ scalars $a_1,a_2,a_3 \in R$ such that

$\alpha=a_1\alpha_1+a_2\alpha_2+a_3\alpha_3$

$(1,0,-1)=a_1(0,1,-1)+a_2(1,1,0)+a_3(1,0,2)$

$(1,0,-1)=(a_2+a_3,\ a_1+a_2,\ -a_1+2a_3)$

$\therefore a_2+a_3=1 \rightarrow (1)$

$a_1+a_2=0 \rightarrow (2)$

$-a_1+2a_3=-1 \rightarrow (3)$

equ (2) and (3).

$$\begin{array}{r} a_1+a_2=0 \\ -a_1+2a_3=-1 \\ \hline a_2+2a_3=-1 \rightarrow (4) \end{array}$$

Solving (1) and (4)

$$\begin{array}{r} a_2+a_3=1 \\ -a_2 \pm 2a_3=-1 \\ \hline -a_3=2 \\ a_3=-2 \end{array}$$

put a_3 in eq (1) $\Rightarrow a_2-2=1$

$a_2=3$

put a_2 in eq (2) $\Rightarrow a_1+3=0$

$a_1=-3$

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$\therefore$ The co-ordinate matrix of $(1,0,-1)$ are $\begin{bmatrix} -3 \\ 3 \\ -2 \end{bmatrix}$

ii) Given $\alpha = (4,5,6)$

Let $S = \{\alpha_1, \alpha_2, \alpha_3\}$

where $\alpha_1 = (1,1,1)$

$\alpha_2 = (-1,1,1)$

$\alpha_3 = (1,0,-1)$

Let there exist scalars $a_1, a_2, a_3 \in \mathbb{R}$ such that

$\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$

$(4,5,6) = a_1(1,1,1) + a_2(-1,1,1) + a_3(1,0,-1)$

$(4,5,6) = (a_1 - a_2 + a_3, a_1 + a_2, a_1 + a_2 - a_3)$

$\therefore a_1 - a_2 + a_3 = 4 \rightarrow (1)$

$a_1 + a_2 = 5 \rightarrow (2)$

$a_1 + a_2 - a_3 = 6 \rightarrow (3)$

Solving (1) and (3)

put a_1 in eq (2) $\Rightarrow 5 + a_2 = 5$

$\Rightarrow a_2 = 0$

put a_1, a_2 in eq (3) $\Rightarrow 5 + 0 - a_3 = 6$

$\Rightarrow a_3 = 6 - 5$

$\Rightarrow a_3 = -1$

$\therefore$ The co-ordinate matrix of $(4,5,6)$ are $\begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}$

iii) Given $\alpha = (2,3,4,-1)$

Let $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$

where $\alpha_1 = (1,1,0,0)$

$\alpha_2 = (0,1,1,0)$

$\alpha_3 = (0,0,1,1)$

$\alpha_4 = (1,0,0,0)$

Let there exist scalars $a_1, a_2, a_3, a_4 \in R$ such that

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4$$

$$(2,3,4,-1) = a_1(1,1,0,0) + a_2(0,1,1,0) + a_3(0,0,1,1) + a_4(1,0,0,0)$$

$$(2,3,4,-1) = (a_1 + a_4, a_1 + a_2, a_2 + a_3, a_3)$$

$\therefore a_1 + a_4 = 2 \rightarrow (1)$

$a_1 + a_2 = 3 \rightarrow (2)$

$a_2 + a_3 = 4 \rightarrow (3)$

$a_3 = -1 \rightarrow (4)$

put $a_3 = -1$ in eq (3) $\Rightarrow a_2 - 1 = 4$

$\Rightarrow a_2 = 5$

put a_2 in eq (2) $\Rightarrow a_1 + 5 = 3$

$\Rightarrow a_1 = 3 - 5$

$\Rightarrow a_1 = -2$

put a_1 in eq (1) $\Rightarrow -2 + a_4 = 2$

$\Rightarrow a_4 = 2 + 2$

$\Rightarrow a_4 = 4$

$\therefore$ The co-ordinates matrix of $(2,3,4,-1)$ are $\begin{bmatrix} -2 \\ 5 \\ -1 \\ 4 \end{bmatrix}$

Quotient Space : If W is a subspace of $V(F)$, then quotient space is denoted by $\frac{V}{W}$, defined by $\frac{V}{W} = \{W + \alpha / \alpha \in V\}$

Q) Let W be a subspace of $V(F)$, then show that $\frac{V}{W}$ is a vector space over 'F': For the vector addition and scalar multiplication are defined by i) $(W+\alpha) + (W+\beta) = W + (\alpha+\beta)$

ii) $a(W+\alpha) = W + a\alpha$, $\forall\ \alpha, \beta \in V(F)$, $a \in F$

Sol:- Given W be a subspace of $V(F)$.

Given set $\frac{V}{W} = \{W + \alpha / \alpha \in V\}$

Given compositions are

i) $(W+\alpha) + (W+\beta) = W + (\alpha+\beta)$ / ii) $a(W+\alpha) = W + a\alpha$, $\forall \alpha, \beta \in V(F), a \in F$

To prove $(\frac{V}{W}, +)$ is a commutative group :

i) Commutative axiom :

$\forall\ \alpha, \beta \in V \Rightarrow \alpha + \beta \in V$

If $\alpha, \beta \in V \Rightarrow \omega + \alpha,\ \omega + \beta \in \frac{V}{W}$

Then $\alpha + \beta$ $(\omega + \alpha) + (\omega + \beta) = \omega + (\alpha + \beta) \in \frac{V}{W}$

$\therefore \frac{V}{W}$ is closed under the given composition addition

ii) Associative axiom :

$\forall\ \alpha, \beta, \gamma \in V \Rightarrow \alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

If $\alpha, \beta, \gamma \in V \Rightarrow (\omega + \alpha), (\omega + \beta), (\omega + \gamma) \in \frac{V}{W}$

consider,

$$(\omega + \alpha) + [(\omega + \beta) + (\omega + \gamma)] = (\omega + \alpha) + [\omega + (\beta + \gamma)]$$
$$= \omega + [\alpha + (\beta + \gamma)]$$
$$= \omega + [(\alpha + \beta) + \gamma]$$
$$= \omega + [(\alpha + \beta) + (\omega + \gamma)]$$
$$= [\omega + (\alpha + \beta)] + (\omega + \gamma)$$

i.e., $(\omega + \alpha) + [(\omega + \beta) + (\omega + \gamma)] = [(\omega + \alpha) + (\omega + \beta)] + (\omega + \gamma)$

$\therefore$ Composition '+' is associative in $\frac{V}{W}$

iii) Identity axiom :

$\forall\ \alpha \in V \Rightarrow \exists$ a vector $\bar{0} \in V$ such that

$a + 0 = a = 0 + a$.

$\alpha + \bar{0} = \alpha = \bar{0} + \alpha$

If $\alpha, \bar{0} \in V \Rightarrow \omega + \alpha,\ \omega + \bar{0} \in \frac{V}{W}$

Consider, $(\omega + \alpha) + (\omega + \bar{0}) = \omega + (\alpha + \bar{0})$

$= \omega + \alpha$

similarly, $(\omega + \bar{0}) + (\omega + \alpha) = \omega + \alpha$.

$\therefore (\omega + \bar{0}) = \omega$ is the identity element in $\frac{V}{W}$

iv) Inverse axiom :

$\forall\ \alpha \in V \Rightarrow \exists$ a vector $-\alpha \in V$ such that

$\alpha+(-\alpha) = (-\alpha)+\alpha = \bar{0}$

If $\alpha, -\alpha \in V \Rightarrow W+\alpha,\ W+(-\alpha) \in \frac{V}{W}$

Consider,

$$(W+\alpha)+(W+(-\alpha)) = W+(\alpha+(-\alpha))$$
$$= W+(\alpha-\alpha)$$
$$= W+0$$
$$= W$$

Similarly $(W+(-\alpha))+(W+\alpha) = W$

$\therefore$ Inverse axiom is satisfied.

v) Commutative axiom :-

$\forall\ \alpha,\beta \in V \Rightarrow \alpha+\beta = \beta+\alpha$

Consider $(W+\alpha)+(W+\beta) = W+(\alpha+\beta)$
$$= W+(\beta+\alpha)$$
$\therefore (W+\alpha)+(W+\beta) = (W+\beta)+(W+\alpha)$

$\therefore$ commutative axiom is satisfied w.r.t composition & addition'

$\therefore \left(\frac{V}{W}, +\right)$ is a commutative group.

ii) Given condition (ii) satisfies external composition

i.e Scalar multiplication of a vector in $\frac{V}{W}$

$\therefore \forall\ a \in F,\ (W+\alpha) \in \frac{V}{W} \Rightarrow a(W+\alpha).$
$$= W+a\alpha \in \frac{V}{W}$$

iii) To prove remaining condition :-

Let $\forall\ a, b \in F$ and $W+\alpha,\ W+\beta \in \frac{V}{W}$

then a) $a[(W+\alpha)+(W+\beta)] = a[(W+(\alpha+\beta))]$
$$= W+a(\alpha+\beta)$$
$$= W+(a\alpha+a\beta)$$
$$= (W+a\alpha)+(W+a\beta)$$

$\therefore a[(W+\alpha)+(W+\beta)] = a(W+\alpha)+a(W+\beta)$

b) $(a+b)\cdot(W+\alpha) = W+(a+b)\alpha$
$$= W+(a\alpha+b\alpha)$$
$$= (W+a\alpha)+(W+b\alpha)$$

$\therefore (a+b)(w+\alpha) = a(w+\alpha) + b(w+\alpha)$

c) $(ab)(w+\alpha) = w + (ab)\alpha$

$= w + a(b\alpha)$

$= a(w + b\alpha)$

$\therefore (ab)(w+\alpha) = a[b(w+\alpha)]$

d) $1 \cdot (w+\alpha) = w + 1 \cdot \alpha$

$\therefore 1(w+\alpha) = w+\alpha$

$\therefore \frac{V}{W}$ is a vector space over the field 'F'.

<u>Note</u> :-

* The zero element of $\frac{V}{W}$ is $0 + w = w$
* $\alpha + w = \beta + w \iff \alpha - \beta \in w$
* $\alpha + w = w \iff \alpha \in w$

<u>Dimension of quotient space</u> :-

Let 'W' be a subspace of a finite dimensional vector space $V(F)$ then $\dim\left(\frac{V}{W}\right) = \dim V - \dim W$

<u>Proof</u>: Given 'W' is a subspace of $V(F)$ where $V(F)$ is finite dimensional

Let $B = \{\alpha_1, \alpha_2, \dots \alpha_m\}$ be a basis of w

$\Rightarrow \dim(w) = m$

If B' is a basis of w

$\Rightarrow$ 'B' is linearly independent and $L(B) = w$

Since 'W' is a subspace of $V(F)$

$\therefore$ 'B' is also linearly independent set in $V(F)$.

$\Rightarrow$ 'B' can be extended to form the basis of $V(F)$

Let $S = \{\alpha_1, \alpha_2, \dots \alpha_m, \beta_1, \beta_2 \dots \beta_l\}$ be a basis of $V(F)$

$\Rightarrow \dim(v) = m + l$.

$\therefore \dim(v) - \dim(w) = \cancel{m} + l - \cancel{m}$

$= l$.

$\Rightarrow \dim(V) - \dim(W) = l$.

Now, we show that $\dim(\frac{V}{W}) = l$:

That is to show that any sub set in $\frac{V}{W}$ containing 'l' elements forms a basis of $\frac{V}{W}$

Let $S' = \{W+\beta_1, W+\beta_2, \dots W+\beta_l\}$ is any subset of $\frac{V}{W}$

To prove that S' is linear independent :

Let $\exists$ scalars $b_1, b_2, \dots b_l \in F$ such that

$b_1(W+\beta_1) + b_2(W+\beta_2) + \dots + b_l(W+\beta_l) = W$ ($\because W$ is the zero element in $\frac{V}{W}$)

$\Rightarrow (W+b_1\beta_1) + (W+b_2\beta_2) + \dots + (W+b_l\beta_l) = W$ ($\because a(W+\alpha) = W+a\alpha$)

$\Rightarrow W + (b_1\beta_1 + b_2\beta_2 + \dots + b_l\beta_l) = W$ ($\because (W+\alpha)+(W+\beta) = W+(\alpha+\beta)$)

$\Rightarrow b_1\beta_1 + b_2\beta_2 + \dots + b_l\beta_l \in W$ ($\because \alpha + W \Leftrightarrow \alpha \in W$)

since 'B' is a basis of W

$\therefore$ Any vector in 'W' can be expressed as a linear combination of vectors present in the basis.

$\therefore \exists$ scalars $a_1, a_2, \dots a_m \in F$ such that

$\Rightarrow b_1\beta_1 + b_2\beta_2 + \dots + b_l\beta_l = a_1\alpha_1 + a_2\alpha_2 + \dots + a_m\alpha_m$

$\Rightarrow b_1\beta_1 + b_2\beta_2 + \dots + b_l\beta_l + (-a_1)\alpha_1 + (-a_2)\alpha_2 + \dots + (a_m)\alpha_m = 0$

since S' is a basis containing α's and β's

$\therefore$ The linear combination of these vectors $= 0$ vector

$\Rightarrow$ All the scalars are zero

$\Rightarrow S'$ is linearly independent

$\therefore b_1 = b_2 = \dots = b_l = a_1 = a_2 = \dots = a_m = 0$

$\because b_1 = b_2 = \dots = b_l = 0$

$\therefore S'$ is linearly independent

To show that $L(S') = \frac{V}{W}$:

Let $\alpha \in V$ be any vector in V, then 'α' can be expressed as a linear combination of vectors present in basis.

$\therefore \exists$ Scalars $c_1, c_2, \dots c_m, d_1, d_2, \dots d_l \in F$ such that

$\alpha = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_m\alpha_m + d_1\beta_1 + \cdots$

$\alpha = \gamma + d_1\beta_1 + d_2\beta_2 + \cdots + d_l\beta_l$

where $\gamma = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_m\alpha_m \in W$

Adding 'w' in the above equation

$w+\alpha = w + (\gamma + d_1\beta_1 + d_2\beta_2 + \cdots + d_l\beta_l)$

$\Rightarrow w+\alpha = (w+\gamma) + (d_1\beta_1 + d_2\beta_2 + \cdots + d_l\beta_l)$

$\Rightarrow w+\alpha = w + (d_1\beta_1 + d_2\beta_2 + \cdots + d_l\beta_l)$ $(\because \gamma \in w)$

$\Rightarrow w+\alpha = (w + d_1\beta_1) + (w + d_2\beta_2) + \cdots + (w + d_l\beta_l)$

$\Rightarrow w+\alpha = d_1(w+\beta_1) + d_2(w+\beta_2) + \cdots + d_l(w+\beta_l) \in L($

$\Rightarrow w+\alpha \in L(s')$

$\therefore$ If $w+\alpha \in \frac{V}{w} \Rightarrow w+\alpha \in L(s')$

$\Rightarrow \frac{V}{w} \subseteq L(s') \rightarrow (1)$

Since 's'' is a subset of $\frac{V}{w}$

& $L(s')$ is also a subset of $\frac{V}{w} \Rightarrow L(s') \subseteq \frac{V}{w} \rightarrow (2)$

$\therefore$ from eq (1) and (2)

$\Rightarrow \frac{V}{w} \subseteq L(s'),\ L(s') \subseteq \frac{V}{w}$

$\Rightarrow L(s') = \frac{V}{w}$

$\therefore$ s' is a basis of $\frac{V}{w}$

$\Rightarrow \dim(\frac{V}{w}) = l$

$\therefore \dim(\frac{V}{w}) = \dim(V) - \dim(w)$

Hence the proof.

linear transformation: Let $U(F)$ and $V(F)$ be two vector spaces over the same field F. The function 'T' : $U \to V$ is said to be "a linear transformation", if $T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$, $\forall \alpha, \beta \in U$ and $a, b \in F$

NOTE

1. In the above definition $T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$ is called linearity property.
2. Let $U(F)$ and $V(F)$ be two vector spaces over the same field F. The mapping $T : U \to V$ is a linear transformation iff i) $T(\alpha + \beta) = T(\alpha) + T(\beta)$
and ii) $T(a\alpha) = aT(\alpha)$, $\forall \alpha, \beta \in V$, $a \in F$

linear Operator: A linear transformation 'T' is said to be "linear operator" if '$T : U(F) \to U(F)$.

Zero transformation: A linear transformation $T : U(F) \to V(F)$ is said to be zero transformation, if $T(\alpha) = \hat{0}$, $\forall \alpha \in U(F)$ where $\hat{0}$ is the zero vector in $V(F)$.

Identity transformation: A linear transformation $T : U(F) \to V(F)$ is said to be identity transformation, if $T(\alpha) = \alpha$, $\forall \alpha \in U(F)$

properties of linear transformation:

Imp **Theorem 23**: Let $T : U(F) \to V(F)$ is a linear transformation from the vector space $U(F)$ to $V(F)$ then

i) $T(\bar{0}) = \hat{0}$

ii) $T(-\alpha) = -T(\alpha)$

iii) $T(\alpha - \beta) = T(\alpha) - T(\beta)$

iv) $T(a_1\alpha_1 + a_2\alpha_2 + \dots + a_m\alpha_m) = a_1T(\alpha_1) + a_2T(\alpha_2) + \dots + a_mT(\alpha_m)$.

Proof: Given, $T : U(F) \to V(F)$ is a linear transformation

$\Rightarrow T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$, $\forall \alpha, \beta \in U(F)$ and $a, b \in F$.

To prove (i): Let $\alpha, \bar{0} \in U(F) \Rightarrow T(\alpha), T(\bar{0}) \in V(F)$

Consider $T(\alpha) + T(\bar{0}) = T(\alpha + \bar{0})$
$= T(\alpha)$
$\therefore T(\alpha) + T(\bar{0}) = T(\alpha) + \hat{0}$

$= T\left(ax_1+by_1,\ ax_2+by_2,\ ax_3+by_3\right)$

$\therefore T(a\alpha+b\beta) = \left(ax_1+by_1-ax_2-by_2,\ ax_1+by_1+ax_3+by_3\right) \to ①$

Again consider $aT(\alpha)+bT(\beta) = aT(x_1,x_2,x_3) + bT(y_1,y_2,y_3)$

$= a(x_1-x_2,\ x_1+x_3) + b(y_1-y_2,\ y_1+y_3)$

$= (ax_1-ax_2,\ ax_1+ax_3) + (by_1-by_2,\ by_1+by_3)$

$= (ax_1-ax_2+by_1-by_2,\ ax_1+ax_3+by_1+by_3)$

$= (ax_1+by_1-ax_2-by_2,\ ax_1+by_1+ax_3+by_3) \to ②$

$\therefore$ From ① & ②, We find that

$T(a\alpha+b\beta) = aT(\alpha)+bT(\beta)$

$\therefore$ T is a linear transformation

2) Test whether $T: R^2 \to R^2$ given by $T(x_1,x_2) = (1+x_1, x_2)$ is a linear transformation.

Sol: Given, $T: R^2 \to R^2$ defined by

$T(x_1,x_2) = (1+x_1, x_2)$

let $\alpha = x_1,x_2,\ \beta = y_1,y_2 \in R^2,\ a,b \in R$.

Consider $T(a\alpha+b\beta) = T\left[a(x_1,x_2)+b(y_1,y_2)\right]$

$= T\left(ax_1+by_1,\ ax_2+by_2\right)$

$T(a\alpha+b\beta) = \left(1+ax_1+by_1,\ ax_2+by_2\right) \to ①$

Again consider

$aT(\alpha)+bT(\beta) = aT(x_1,x_2)+bT(y_1,y_2)$

$= a(1+x_1,x_2)+b(1+y_1,y_2)$

$= (a+ax_1,\ ax_2)+(b+by_1,\ by_2)$

$\therefore aT(\alpha)+b(T\beta) = (a+ax_1+b+by_1,\ ax_2+by_2) \to ②$

$\therefore$ From equ ① & ②, We find that

$T(a\alpha+b\beta) \neq aT(\alpha)+bT(\beta)$

$\therefore$ 'T' is not a linear transformation.

3) S.T the function $T: V_2(R) \to V_2(R)$ defined by $T(x,y) = (0,y)$ is a linear transformation.

4) S.T the transformation $T: R^3 \to R^3$ defined by $T(x,y,z) = (x-y,\ 0,\ y+z)$ is a linear transformation.

Where $\hat{0}$ is the zero element in $V(F)$

$\therefore T(\alpha) + T(0) = T(\alpha) + \hat{0}$

$\Rightarrow T(0) = \hat{0}$ [$\because$ left cancellation law]

To prove (ii):

Consider $T(-\alpha) = T[(-1)\alpha]$

$= (-1)\, T(\alpha)$ $\quad \because T(a\alpha) = aT(\alpha)$

$\therefore T(-\alpha) = -T(\alpha)$

To prove (iii):

Consider $T(\alpha - \beta) = T[\alpha + (-\beta)]$

$= T(\alpha) + T(-\beta)$

$\therefore T(\alpha - \beta) = T(\alpha) - T(\beta)$

To prove (iv): To prove the result, We applyed induction meth

If $m=1$ then $T(a_1\alpha_1) = a_1 T(\alpha_1)$

If $m=2$ then $T(a_1\alpha_1 + a_2\alpha_2) = a_1 T(\alpha_1) + a_2 T(\alpha_2)$

$\therefore$ The result is ~~trave~~ true for $m=1, m=2$

let us assume that the result is true for $m=k$

Prove the result for $m=k+1$

Consider $T[(a_1\alpha_1 + a_2\alpha_2 + \dots + a_k\alpha_k) + a_{k+1}\alpha_{k+1}] =$

$T(a_1\alpha_1 + a_2\alpha_2 + \quad + a_k\alpha_k) + T(a_k + \alpha_{k+1})$

$= a_1 T(\alpha_1) + a_2 T(\alpha_2) + \dots + a_k T(\alpha_k) + a_{k+1} T(\alpha_{k+1})$

$\therefore$ The result is true for $m=k+1$

$\therefore T(a_1\alpha_1 + a_2\alpha_2 + \dots + a_m\alpha_m) = a_1 T(\alpha_1) + a_2 T(\alpha_2) + \dots + a_m T(\alpha_n)$

Hence the proof

1) If 'T' is mapping

1) If $T: V_3(R) \to V_2(R)$ is defined as $T(x_1, x_2, x_3) = (x_1 - x_2, x$

Prove that T is a linear transformation.

Sol: Given, $T: V_3(R) \to V_2(R)$ defined by $T(x_1, x_2, x_3) = (x_1 - x_2, x$

let $\alpha = (x_1, x_2, x_3)$, $\beta = (y_1, y_2, y_3) \in V_3(R)$ and $a, b \in R$

Consider, $T(a\alpha + b\beta) = T[a(x_1, x_2, x_3) + b(y_1, y_2, y_3)]$

the mapping $T: V_3(R) \to V_2(R)$ defined $T(a_1, a_2, a_3) = (3a_1 - 2a_2 + a_3, a_1 - 3a_2 - 2a_3)$ is a linear transformation from $V_3(R)$ to $V_2(R)$

c) S.T. the mapping $T: R^2 \to R^3$ defined by $T(a,b) = (a+b, a)$ is a linear transformation

7) S.T. $T: V_3(R) \to V_3(R)$ defined by $T(a,b,c) = a^2 + b^2 \ldots$ is not a linear transformation.

3) Sol: Given, $T: V_2(R) \to V_2(R)$

$$T(x,y) = (0,y)$$

Let $\alpha = (x_1, y_1)$, $\beta = (x_2, y_2) \in R$

consider, $T(a\alpha + b\beta) = T[a(x_1, y_1) + b(x_2, y_2)]$

$$= T[(ax_1 + bx_2), (ay_1 + by_2)]$$

$$\therefore T(a\alpha + b\beta) = [0, ay_1 + by_2] \to ①$$

Again consider

$$aT(\alpha) + bT(\beta) = aT(x_1, y_1) + bT(x_2, y_2)$$

$$= a(0, y_1) + b(0, y_2)$$

$$= (0, ay_1) + (0, by_2)$$

$$= (0, ay_1 + by_2) \to ②$$

$\therefore aT(\alpha) + bT(\beta)$

from ① and ② we find that

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$$

$\therefore$ 'T' is a linear transformation

4) Sol: Given $T: R^2 \to R^3$ defined by $T(x,y,z) = (x-y, 0, y+z)$

let $\alpha = (x_1, y_1, z_1)$ $\beta = (x_2, y_2, z_2) \in R^3$

consider

$$T(a\alpha + b\beta) = T[a(x_1, y_1, z_1) + b(x_2, y_2, z_2)]$$

$$= T[(ax_1 + bx_2), (ay_1 + by_2), (az_1 + az_2)]$$

$$\therefore T(a\alpha + b\beta) = (ax_1 + bx_2 - ay_1 + by_2, 0, ay_1 + by_2 + az_1 + bz_2) -$$

Again consider

$$aT(\alpha)+bT(\beta) = aT(x_1,y_1,z_1) + bT(x_2,y_2,z_2)$$

$$= a(x_1-y_1, 0, y_1+z_1) + b(x_2-y_2, 0, y_2+z_2)$$

$$= (ax_1-ay_1, 0, ay_1+az_1) + (bx_2-by_2, 0, by_2+bz_2)$$

$$= (ax_1-ay_1+bx_2-by_2, 0, ay_1+az_1+by_2+bz_2)$$

$$\therefore aT(\alpha)+bT(\beta) = (ax_1+bx_2-ay_1-by_2, 0, ay_1+by_2+az_1+bz_2) \rightarrow (2)$$

From (1) & (2), We find that

$$T(a\alpha+b\beta) = aT(\alpha)+bT(\beta)$$

$\therefore$ 'T' is a linear transformation.

6) sol: Given $T: V_3(R) \rightarrow V_2(R)$ defined by

$$T(a_1,a_2,a_3) = (3a_1-2a_2+a_3,\ a_1-3a_2-2a_3)$$

let $\alpha = (a_1,a_2,a_3)$, $\beta = (b_1,b_2,b_3) \in V_3(R)$ and $a,b \in R$

consider

$$T(a\alpha+b\beta) = T[a(a_1,a_2,a_3) + b(b_1,b_2,b_3)]$$

$$= T[aa_1+bb_1,\ aa_2+bb_2,\ aa_3+bb_3]$$

$$= [3(aa_1+bb_1) - 2(aa_2+bb_2) + aa_3+bb_3,\ aa_1+bb_1-3aa_2+3bb_2-2aa_3+2bb_3]$$

$$\therefore T(a\alpha+b\beta) = [3aa_1+3bb_1-2aa_2-2bb_2+aa_3+bb_3,\ aa_1+bb_1-3aa_2+3bb_2-2aa_3+2bb_3] \rightarrow (1)$$

Again consider

$$aT(\alpha)+bT(\beta) = aT(a_1,a_2,a_3) + bT(b_1,b_2,b_3)$$

$$= a[3a_1-2a_2+a_3,\ a_1-3a_2-2a_3] + b[3b_1-2b_2+b_3,\ b_1-3b_2-2b_3]$$

$$= [3aa_1-2aa_2+aa_3,\ aa_1-3aa_2-2aa_3+3bb_1-2bb_2+bb_3,\ bb_1-3bb_2-2bb_3]$$

$$\therefore aT(\alpha)+bT(\beta) = [3aa_1+3bb_1-2aa_2-2bb_2+aa_3+bb_3,\ aa_1+bb_1-3aa_2+3bb_2-2aa_3+2bb_3] \rightarrow (2)$$

From eqn (1) and (2) we find that

$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$

$\therefore T$ is a linear transformation

3) Sol: Given $T : R^2 \to R^3$ defined by $T(a, b) = (a + b, a)$

let $\alpha = (a_1, b_1)$, $\beta = (a_2, b_2)$

Consider

$T(a\alpha + b\beta) = T[a(a_1, b_1) + b(a_2, b_2)]$

$= T[aa_1 + ba_2, ab_1 + bb_2]$

$T(a\alpha + b\beta) = (aa_1 + ba_2 + ab_1 + bb_2, aa_1 + ba_2)$ — (1)

Again consider

$aT(\alpha) + bT(\beta) = aT(a_1, b_1) + bT(a_2, b_2)$

$= a(a_1 + b_1, a_1) + b(a_2 + b_2, a_2)$

$[aa_1 + ab_1, aa_1 + ba_2 + bb_2, ba_2]$

$aT(\alpha) + bT(\beta) = (aa_1 + ba_2 + ab_1 + bb_2, aa_1 + ba_2)$ — (2)

From eqn (1) and (2) we find that

$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$

$\therefore T$ is a linear transformation

4) Sol: Given $T : V_3(R) \to V_1(R)$ defined by

$T(a, b, c) = a^2 + b^2 + c^2$

let $\alpha = (a_1, b_1, c_1)$, $\beta = (a_2, b_2, c_2) \in V_3(R)$ and $a, b \in R$

consider

$T(a\alpha + b\beta) = T[a(a_1, b_1, c_1) + b(a_2, b_2, c_2)]$

$= T(aa_1 + ba_2, ab_1 + bb_2, ac_1 + bc_2)$

$= T(aa_1 + ba_2, ab_1 + bb_2, ac_1 + bc_2)$

$\therefore T(a\alpha + b\beta) = (aa_1 + ba_2)^2 + (ab_1 + bb_2)^2 + (ac_1 + bc_2)^2$ — (1)

Again consider

$aT(\alpha) + bT(\beta) = aT(a_1, b_1, c_1) + bT(a_2, b_2, c_2)$

$= a(a_1^2 + b_1^2 + c_1^2) + b(a_2^2 + b_2^2 + c_2^2)$

$= (aa_1^2 + ab_1^2 + ac_1^2) + (ba_1^2 + bb_1^2 + bc_1^2)$

$\therefore aT(\alpha) + bT(\beta) = (aa_1^2 + ab_1^2 + ac_1^2) + (ba_1^2 + bb_1^2 + bc_1^2) \rightarrow (3)$

From equ (1) and (3), we find that

$T(a\alpha + b\beta) \neq aT(\alpha) + bT(\beta)$

$\therefore$ 'T' is not a linear transformation.

5) verify $T : R^3 \to R^3$ defined by $T(x, y, z) = (|x|, 0)$ is a linear transformation.

Sol: Given, $T : R^3 \to R^3$ defined by

$T(x, y, z) = (|x|, 0)$

let $\alpha = (x_1, y_1, z_1)$ $\beta = (x_2, y_2, z_2) \in R^3$ and $a, b \in R$

consider

$T(a\alpha + b\beta) = T[a(x_1, y_1, z_1) + b(x_2, y_2, z_2)]$

$= T[ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2]$

$\therefore T(a\alpha + b\beta) = (|ax_1 + bx_2|, 0) \rightarrow (1)$

Again consider

$aT(\alpha) + bT(\beta) = aT(x_1, y_1, z_1) + bT(x_2, y_2, z_2)$

$= a(|x_1|, 0) + b(|x_2|, 0)$

$= (a|x_1|, 0) + (b|x_2|, 0)$

$\therefore aT(\alpha) + bT(\beta) = (a|x_1| + b|x_2|, 0) \rightarrow (2)$

From (1) and (2)

$T(a\alpha + b\beta) \neq aT(\alpha) + bT(\beta)$

$\therefore$ 'T' is not a linear transformation.

Determination of linear transformation:

Theorem 24: Let $U(F)$ and $V(F)$ be two vector spaces and $S = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ be a basis of $U(F)$. Let $\{\delta_1, \delta_2 \ldots \delta_n\}$ be a set of vectors in $V(F)$ then there exist a unique linear transformation $T : U(F) \to V(F)$ such that $T(\alpha_i) = \delta_i$, for $i = 1, 2 \ldots n$

Proof: Given, $U(F)$ and $V(F)$ are any two vector spaces also given $S = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ be a basis of $U(F)$

$\Rightarrow$ any vector in $U(F)$ can be expressed as a linear combination of vector in the basis 'S'.

$\therefore$ If $\alpha \in U \Rightarrow$ there exist scalars $a_1, a_2 \ldots a_n \in F$ such that

$\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$

If the set $\{\delta_1, \delta_2 \ldots \delta_n\}$ is linearly independent in $V(F)$

$\Rightarrow$ there exist scalars $a_1, a_2 \ldots a_n \in F$ such that

$$a_1\delta_1 + a_2\delta_2 + \cdots + a_n\delta_n = \bar{0}$$

Let $T : U(F) \longrightarrow V(F)$ is any function defined by

$$T(\alpha) = T(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n)$$
$$= a_1\delta_1 + a_2\delta_2 + \cdots + a_n\delta_n$$

We can write $\alpha_i = 0\alpha_1 + 0\alpha_2 + \cdots + 0\alpha_{i-1} + \alpha_i + 0\alpha_{i+1} + \cdots$

$\Rightarrow T(\alpha_i) = 0\delta_1 + 0\delta_2 + \cdots + 0\delta_{i-1} + \delta_i + 0\delta_{i+1} + \cdots + 0\delta_n$

$\Rightarrow T(\alpha_i) = \delta_i$

$\therefore T(\alpha_i) = \delta_i$, for $i = 1, 2 \ldots n$

$\therefore$ T exist

To prove 'T' is a linear transformation:

If $\alpha, \beta \in U(F) \Rightarrow$ there exist scalars $a_1, a_2 \ldots a_n, b_1, b_2 \ldots b_n \in F$ such that $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$

and $\beta = b_1\bar{\beta}_1 + b_2\bar{\beta}_2 + \cdots + b_n\bar{\beta}_n$

consider

$$T(a\alpha + b\beta) = T[a(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) + b(b_1\beta_1 + b_2\beta_2 + \cdots + b_n\beta_n)]$$
$$= T[aa_1\alpha_1 + aa_2\alpha_2 + \cdots + aa_n\alpha_n + bb_1\bar{\beta}_1 + bb_2\bar{\beta}_2 + \cdots + bb_n\beta_n]$$
$$= T[(aa_1 + bb_1)\alpha_1 + (aa_2 + bb_2)\alpha_2 + \cdots + (aa_n + bb_n)\alpha_n]$$
$$= (aa_1 + bb_1)\delta_1 + (aa_2 + bb_2)\delta_2 + \cdots + (aa_n + bb_n)\delta_n$$
$$= a(a_1\delta_1 + a_2\delta_2 + \cdots + a_n\delta_n) + b(b_1\delta_1 + b_2\delta_2 + \cdots + b_n\delta_n)$$

$\Rightarrow T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$, $\forall\ \alpha, \beta \in U(F)$

$\therefore$ ~~$T(a\alpha+b\beta)$~~ 'T' is a linear transformation

To prove 'T' is unique:

If possible, let us assume that there exist a linear transformation $T : U(F) \rightarrow V(F)$ such that $T'(\alpha_i) = \delta_i$, $i = 1, 2 \ldots n$